Par-hexagons
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Convex quadrilaterals with their opposite sides parallel have been very well understood for a long time. The situation for convex polygons with more than four sides has received less attention. In [2] we examined the case of pentagons and defined a par-pentagon to be a convex pentagon for which each side is parallel to its opposite diagonal. This turns out to be a rather strong condition and up to affine equivalence there is just one par-pentagon, represented by the regular pentagon. In this article we examine the hexagon case and we define a par-hexagon to be a convex hexagon for which each side is parallel to its opposite side. As expected, this turns out to be a much weaker condition than the one for pentagons; up to affine equivalence the family of par-hexagons is essentially parametrized by one vector and one ratio. We know only a few general properties of par-hexagons, but there are several interesting subfamilies of par-hexagons. We confine our attention to convex hexagons here; there are rather many non-convex hexagons on six points.

An intuitive way to produce par-hexagons is to start with any parallelogram and then cut off two corner triangles by parallel lines. Or we may start with a triangle and cut off from the corners three triangles which are similar to each other. Or we may take the intersection of two similar triangles placed suitably as in the third diagram above. Another characterization of par-hexagons is that each pair of diagonally opposite angles should be equal.

EXERCISE 1. Prove that a hexagon is a par-hexagon if and only if diagonally opposite angles are equal.

Each of the above viewpoints will appear in diagrams for proofs, but it turns out to be better to make a systematic use of vectors. Let \( \Pi \) be any
convex hexagon, and let three successive sides represent the vectors $a, b, c$. For a par-hexagon the fourth side must represent the vector $-\lambda a$ for some $\lambda > 0$. For a par-hexagon the fifth and sixth sides are now uniquely determined by the remaining two parallel conditions. Thus the complete family of par-hexagons is parametrized by three vectors and one ratio (modulo the convexity requirement). In terms of vectors, a par-hexagon is described by $a, b, c, -\lambda a, -\mu b, -\nu c$ with $\lambda, \mu, \nu > 0$ and

$$(1 - \lambda)a + (1 - \mu)b + (1 - \nu)c = 0$$

If $\lambda = 1$, it follows by the linear independence of $b, c$ that we also have $\mu = 1 = \nu$. (This is also clear since the parallelogram with vector sides $a$ is bordered by two congruent triangles.) We call these isosceles par-hexagons. Within this subfamily we have the equilateral par-hexagons for which all six sides have the same length. For the general non-isosceles case we may suppose without loss of generality that $0 < \lambda < 1$. We cannot have $\mu < 1$ since this would contradict the convexity requirement in that the vector $(1 - \lambda)a + (1 - \mu)b$ would lie in the first quadrant with respect to the basis vectors $a, b$. So we must have $\mu > 1$ and a similar argument then gives $0 < \nu < 1$. We obtain another special family of par-hexagons by the condition that $\lambda = \frac{1}{\mu} = \nu$. We call these equiratio par-hexagons, and we include the isosceles par-hexagons in this family. Examples of these appeared in [2] and we give other examples later. Another interesting special family (which appears in Exercises in Coxeter and Greitzer [1]) has the property that each diagonal is parallel to the remaining pair of opposite sides; we call these tripar-hexagons. Yet another interesting family is given by the cyclic par-hexagons.

We now give two general properties of par-hexagons. By a median of
a hexagon we mean the line joining the mid-points of opposite sides. By the center of gravity of a hexagon we mean the center of gravity of the six vertices of the hexagon. We use $\wedge$ for vector product. We denote the area of a convex polygon $C$ by $|C|$.

**THEOREM 1.** Let $\Pi$ be any par-hexagon. Then the medians of $\Pi$ are concurrent.

Proof. Let $\Pi$ be given by $ABCDEF$ and let $P,Q,R,S,T,U$ be the mid-points of the six sides of $\Pi$ in order, starting with $AB$. Let $PS, QT$ meet at $O$. Let $a, b, c, d, e, f$ be the vectors represented by the arrows from $O$ to the six vertices of $\Pi$ in order. Since opposite sides of $\Pi$ are parallel we have

$$ (a - b) \wedge (d - e) = 0 \quad (1) $$

$$ (b - c) \wedge (e - f) = 0 \quad (2) $$

$$ (c - d) \wedge (f - a) = 0 \quad (3) $$

Since $P, O, S$ and $Q, O, T$ are collinear, we also have

$$ (a + b) \wedge (d + e) = 0 \quad (4) $$

$$ (b + c) \wedge (e + f) = 0 \quad (5) $$

Let

$$ z = (c + d) \wedge (f + a) \quad (6) $$

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We show that $z = 0$, and so the medians of $\Pi$ are concurrent. From (1) and (4) we have
\[ b \wedge e + a \wedge d = 0 \quad (7) \]
From (2) and (5) we have
\[ b \wedge e + c \wedge f = 0 \quad (8) \]
From (3) and (6) we have
\[ a \wedge d + c \wedge f = \frac{1}{2}z \quad (9) \]
From (7), (8), (9) we find that $\frac{1}{2}z = a \wedge d + d \wedge a = 0$ as required.

THEOREM 2. Let $\Pi = ABCDEF$ be any par-hexagon. Then triangles $ACE, BDF$ have the same area.

Proof. We leave the reader to provide a proof by the use of vector products. The following pure proof is more appealing. Extend the picture of $\Pi$ in two ways as in the diagram. Thus we have parallels:

\[ AB \parallel ED \parallel PC \parallel FT, \quad BC \parallel FE \parallel AR \parallel SD, \quad CD \parallel AF \parallel QE \parallel BU \]

The left diagram gives
\[ |ACE| = \frac{1}{2}|ABCP| + \frac{1}{2}|CDEQ| + \frac{1}{2}|EFAR| + |PQR| \]
and so $|ACE| = \frac{1}{2}(|\Pi| + |PQR|)$. Similarly, from the right diagram, $|BDF| = \frac{1}{2}(|\Pi| + |STU|)$. We easily see that triangles $PQR, STU$ are congruent by (SSS), and so $|ADC| = |BDF|$.
We now consider the family of equiratio par-hexagons. Around the hexagon the vectors are given by \(a, b, c, -\lambda a, \frac{1}{\lambda}b, -\lambda c\) and they satisfy the equation

\[(1 - \lambda)a + (1 - \frac{1}{\lambda})b + (1 - \lambda)c = 0\]  

\((*)\)

Suppose first that \(0 < \lambda < 1\). Then equation \((*)\) is equivalent to the equation

\[a + c = \frac{1}{\lambda}b\]  

\((**)\)

Then \(AD\) has vector \(a + b + c = (1 + \frac{1}{\lambda})b\) and so \(AD \parallel BC \parallel FE\). Also, \(BE\) has vector \(b + c - \lambda a = (1 + \lambda)c\) so that \(BE \parallel CD \parallel AF\). Similarly \(FC \parallel AB \parallel ED\). Thus \(ABCDEF\) is a tripar-hexagon. Equation \((*)\) gives no information when \(\lambda = 1\). But if equation \((**)\) holds with \(\lambda = 1\), then \(a + c = b\) and the above argument shows that we have a tripar-hexagon. Conversely, for any tripar-hexagon, \(AD\) has vector \(kb\) for some \(k > 1\) and we obtain \(a+c = (k-1)b\). Thus equation \((**)\) characterizes tri-par hexagons for \(0 < \lambda \leq 1\). Any three vectors \(a, b, c\) determine an isosceles par-hexagon, but only a very special subfamily of these are tripar-hexagons. In this connection, it is worth to repeat the very nice result and proof of an exercise in Coxeter and Greitzer [1, page 73, Exercise 1].

**PROPOSITION 1.** Let \(\Pi = ABCDEF\) be a par-hexagon with \(AD \parallel BC\) and \(BE \parallel CD\). Then \(CF \parallel AB\).

**Proof.** Extend the lines \(AB, CD, EF\) so as to form a triangle \(UVW\) with \(A\) and \(B\) on \(UV\), \(C\) and \(D\) on \(VW\), \(E\) and \(F\) on \(WU\). Since \(UE = AD = FW\), we have \(UF = EW = BC\). Thus \(BCFU\) is a parallelogram and \(CF\) is parallel to \(AB\).

**EXERCISE 2.** Let \(\Pi = ABCDEF\) be a tripar-hexagon. Give a vector proof that triangles \(ACE, BDF\) have the same centroid.
We saw in Theorem 1 that the medians of any par-hexagon are concurrent. For special par-hexagons we can identify the point of concurrence.

PROPOSITION 2. Let $Π = ABCDEF$ be any isosceles par-hexagon. Then the diagonals and the medians of $Π$ all concur at the center of gravity of $Π$.

Proof. Since $ABDE$ is a parallelogram, $AD$, $BE$ intersect at their mid-points. Similarly for $BE$, $CF$. Thus the three diagonals are concurrent. Clearly each median passes through this point of concurrence, which is easily seen to be the center of gravity of $Π$.

PROPOSITION 3. Let $Π = ABCDEF$ be a tripar-hexagon. Then the medians of $Π$ concur at the center of gravity of $Π$.

Proof. We use the notation in the proof of Theorem 1. We know that the medians of $Π$ concur at $O$ and so it remains to show that $O$ is the center of gravity of $Π$. Extend $CD$ and $EF$ to meet at $X$, and $CB$ and $FA$ to meet at $Y$. Then the tripar condition ensures that $XY$ passes through $S$, $P^*$ and $P$, where $P^*$ is the mid-point of $QU$. Thus the median $SP$ of $Π$ includes the median $SP^*$ of triangle $SUQ (=Δ, say)$. Similarly for the other medians so that $O$ must be the centroid of $Δ$. But the centroid of $Δ$ has position vector
\[ \frac{1}{3} [\frac{1}{2} (a + b) + \frac{1}{2} (c + d) + \frac{1}{2} (e + f)] = \frac{1}{6} (a + b + c + d + e + f) \]
which is the center of gravity of $Π$.

PROPOSITION 4. Let $Π = ABCDEF$ be a par-hexagon with $P, Q, R, S, T, U$ the mid-points of the sides in order starting with $AB$. Let $Π$ be such that the medians of $Π$ concur at the center of gravity of $Π$. Then $|PO| : |OS| = |RO| : |OU| = |TO| : |OQ|$ and $Π$ is either isosceles or tripar.

Proof. We again use the notation and diagram for Theorem 1. Note first that we cannot have $U, O, Q$ collinear, and hence $f + a$ and $b + c$ are linearly independent. For some positive scalars $s, t, u$ we have
\[ a + b = -s(d + e), \quad c + d = -t(f + a), \quad e + f = -u(b + c) \]
and so
\[ s(d + e) + t(f + a) + u(b + c) = -(a + b + c + d + e + f) = 0 \]
Also

\[ s(d + e) + s(f + a) + s(b + c) = s(d + e + f + a + b + c) = 0 \]

It follows that \((t - s)(f + a) + (u - s)(b + c) = 0\) and so \(s = t = u\) since \(f + a\) and \(b + c\) are linearly independent. We now have \(a + b = -t(d + e)\) and \(e + f = -t(b + c)\), so that \(a + td = -b - te\) and \(f + tc = -e - tb\). It follows that \((f - a) + t(c - d) = (1 - t)(b - e)\). If \(t = 1\), we have \(|FA| = |CD|\), and similarly for the other opposite pairs of sides, so that \(\Pi\) is isosceles. If \(t \neq 1\), then, since \(FA\) is parallel to \(DC\), they are both parallel to \(EB\), and similarly for the other two principal diagonals, so that \(\Pi\) is a tripar-hexagon.

We now give another family of equiratio par-hexagons. Let \(ABC\) be any triangle with medians \(AD, BE, CF\) meeting at \(G\). Let the centroids of triangles \(GBD, GDC, GCE, GEA, GAF, GFB\) be \(P, Q, R, S, T, U\) respectively. It is a routine exercise on similar triangles to show that \(PQRSTU\) is a par-hexagon. Again by similar triangles we have \(TS = \frac{2}{3}ML = \frac{1}{3}BC\) and \(PQ = \frac{2}{3}HJ = \frac{11}{32}BC\) so that \(PQ = 2TS\). Similarly, \(RS = 2PU\) and \(TU = 2RQ\), so that \(PQRSTU\) is an equiratio par-hexagon. (It is also a well known and interesting exercise to show that \(|PQRSTU| = \frac{13}{36}|ABC|\).)
Conversely, let \( \Pi = PQRSTU \) be any equiratio par-hexagon with \( PQ = 2TS \). We show that there is a triangle \( ABC \) for which \( PQRSTU \) is the par-hexagon of centroids as above. Let sides \( PQ, QR, RS \) correspond to vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) respectively. As noted above we have \( \mathbf{a} + \mathbf{c} = 2\mathbf{b} \). It is elementary that the plane may be tiled (as in the diagram) with congruent triangles, each with vector sides \( \frac{1}{2}\mathbf{a}, \frac{1}{2}\mathbf{c}, \mathbf{b} \). The diagram shows the tiling for triangle \( XYZ \) with \( \Pi \) within the triangle. Let \( X', Y', Z' \) be the mid-points of \( YZ, ZX, XY \) respectively. Since \( UP \parallel XZ \), the median \( YY' \) passes through the mid-point of \( UP \) and hence through the center of gravity \( G \) of \( \Pi \). Similarly for the other two medians, so that \( G \) is the centroid of triangle \( XYZ \). Extend the three medians so that \( GX:XA = GY:YB = GZ:ZC = 2:1 \). Clearly triangle \( ABC \) is similar to triangle \( XYZ \) and the medians of triangle \( XYZ \) extend to give the medians of triangle \( ABC \) which also has centroid at \( G \). Let \( GP \) meet \( BC \) at \( H \). Then \( H \) is the mid-point of \( BD \) and \( GP:PH = 2:1 \). Hence \( P \) is the centroid of triangle \( GPD \). The same argument applies around triangle \( ABC \) and so the proof is complete.

**EXERCISE 6.** Let \( \Pi \) be an equiratio par-hexagon with opposite side ratio 1:t. Show that the center of gravity divides each median in the ratio
EXERCISE 7. Let $\Pi = ABCDEF$ be equiratio with $|AB| = t|DE|$. Show that

(a) the perpendicular bisectors of $AB, CD$ and $EF$ meet at $I$ (say),
(b) the perpendicular bisectors of $BC, DE$ and $FA$ meet at $J$ (say),
(c) $I, O, J$ are collinear ($O$ being the center of gravity of $\Pi$),
(d) $O$ divides $IJ$ in the ratio $(2 + t) : (1 + 2t)$ [the same ratio as $O$ divides the medians].

We now return to the issue of the parametrization of three special families of par-hexagons. For the equiratio family we start with vectors $a, b$. As soon as we know the common ratio $\lambda$ (where $0 < \lambda \leq 1$) we know the third vector from $a + c = \frac{1}{\lambda}b$ and hence we know the whole par-hexagon. Thus, up to affine equivalence (and modulo the convexity requirement) the equiratio family is parametrized by just the ratio $\lambda$. By contrast, the isosceles family requires us to know the first three vectors $a, b, c$ (modulo the convexity requirement), and so up to affine equivalence the isosceles family is parametrized by one vector. This apparent paradox is explained by the fact that, in the equiratio family with $0 < \lambda < 1$, every diagonal is parallel to the associated opposite sides, a property not shared by a generic isosceles par-hexagon. The isosceles family has another nice description. Given isosceles par-hexagon $\Pi = ABCDEF$, construct $O$ inside $\Pi$ so that $AO \parallel BC$ and $CO \parallel DE$. It is elementary that $OE \parallel CD$ and so $\Pi$ is dissected into three parallelograms. Conversely, given line segments $OA, OC, OE$ we may complete three parallelograms to obtain an isosceles par-hexagon $ABCDEF$. Finally, consider the equilateral family. The above construction for isosceles par-hexagons applies except that we obtain three congruent rhombuses in this case. Conversely, three line segments with equal length lead to an equilateral par-hexagon. (The notion of affine equivalence does not apply to this family.)

We now consider very briefly two more special families of par-hexagons. A par-hexagon is cyclic if the six vertices lie on a circle; a par-hexagon is cocyclic if the six sides are tangent to a circle. Let $\Pi = ABCDEF$ be a cyclic hexagon. By Exercise 1, $\Pi$ is a par-hexagon if and only if opposite angles of $\Pi$ are equal. This is easily seen to be equivalent to the condition that triangle $ACE$ is similar to triangle $DFB$ (or congruent, since they lie in the same
circle). So the family of cyclic par-hexagons is obtained as follows. Draw any triangle $ACE$ in a circle. Rotate the triangle (in the positive direction) so that $A$ passes $C$ but not $E$, and label the rotated triangle as $DFB$. Then $ABCDEF$ is a cyclic par-hexagon.

EXERCISE 8. Let $\Pi = ABCDEF$ be a cyclic hexagon and let the sides (starting with $AB$) subtend angles $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ at the center of the circle. Then $\Pi$ is a par-hexagon if and only if

$$\delta - \alpha = \beta - \epsilon = \zeta - \gamma$$

EXERCISE 9. An isosceles cyclic par-hexagon can have just one of its diagonals parallel to the corresponding opposite sides.

EXERCISE 10. A cyclic par-hexagon is tripar if and only if it is a regular hexagon.

EXERCISE 11. The par-hexagon from the six centroids in a triangle is cyclic if and only if the triangle is equilateral.

EXERCISE 12. Let $\Pi$ be a par-hexagon. Show that $\Pi$ is co-cyclic if and only if $\Pi$ is isosceles and the three pairs of opposite sides are the same (perpendicular) distance apart.

Given any convex $n$-gon $\Pi = A_1A_2A_3 \ldots A_n$, (with $n \geq 5$), we may construct two associated convex $n$-gons. Draw the $n$ alternate chords $A_1A_3, A_2A_4, \ldots$ and these bound an inner convex $n$-gon which we denote by $I\Pi$. Extend each side and let alternate pairs of sides meet in $n$ points to give an external convex $n$-gon which we denote by $O\Pi$. It is easy to see that invariance by $I$ or $O$ fails except on the family of isosceles par-hexagons. Let $\Pi$ be any isosceles par-hexagon with center of gravity $G$. We leave as an exercise the fact that each of $I\Pi, O\Pi$ has center of gravity at $G$. It appears to be true that $I^n\Pi$ shrinks to the point $G$.
