Par-pentagons

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Convex quadrilaterals with opposite sides parallel are very well understood in Euclidean geometry; and in affine geometry they are all equivalent to the unit square. For convex pentagons there is no natural notion of opposite side, but there is a natural pairing of a side and the opposite diagonal (which is a side of the inscribed pentagram). We define a par-pentagon to be a convex pentagon for which each side is parallel to its opposite diagonal. In fact we show that it is enough to have the parallel condition true for four sides of the convex pentagon. We present various characterizations of par-pentagons and show, in particular, that each is affine equivalent to a regular pentagon. We employ a combination of vector proofs and pure proofs. Finally we show that there is little to add to the story when we consider the parallel condition in a non-convex setting.

Since Golden Ratios appear frequently for par-pentagons, we set up our notation for these. We write $G = (\sqrt{5} + 1)/2$, $g = (\sqrt{5} - 1)/2$ for the usual Golden Ratios, so that $G^2 = G + 1$, $g^2 + g = 1$, $G = 1/g$, $G - g = 1$. We denote the length of a line (segment) $XY$ by $|XY|$ and the area of a triangle $XYZ$ by $|XYZ|$.

THEOREM 1. Let $ABCDE$ be a convex pentagon with

\[ AB \parallel EC, \quad BC \parallel AD, \quad CD \parallel BE, \quad DE \parallel CA. \]

Then $EA \parallel DB$, so that $ABCDE$ is a par-pentagon. Also, the ratio of each diagonal to its opposite side is the Golden Ratio $G$.

Proof. Let $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EA}$ represent the vectors $a, b, c, d, e$ respectively. Then, for some positive reals $\lambda, \mu, \nu, \rho$, we have

\[ c + d = -\lambda a, \quad d + e = -\mu b, \quad e + a = -\nu c, \quad a + b = -\rho d. \]

It follows that

\[ \rho c = -\rho \lambda a + a + b, \quad \rho e = -\rho \mu b + a + b. \]

Since $a + b + c + d + e = 0$, substitution leads to

\[ (1 + \rho - \rho \lambda)a + (1 + \rho - \rho \mu)b = 0. \]
Since \( \mathbf{a}, \mathbf{b} \) are linearly independent, we obtain \( \rho \lambda = 1 + \rho = \rho \mu \) and hence \( \lambda = \mu \). Now substitute in the equation \( \mathbf{e} + \mathbf{a} = -\nu \mathbf{c} \) to obtain

\[
\mathbf{a} + (1 - \lambda \rho) \mathbf{b} + \rho \mathbf{a} = (\rho \lambda \nu - \nu) \mathbf{a} - \nu \mathbf{b}.
\]

But \( 1 - \lambda \rho = -\rho \) and so \( \nu = \rho \). We also have \( \rho = \lambda \rho - 1 \). This gives \( 1 + \rho = \rho \nu = \rho^2 \), \( \rho = (\sqrt{5} + 1)/2 = G \). Finally \( \lambda = 1 + 1/\rho = \rho \), and so \( \lambda = \mu = \nu = \rho \). Add the four initial vector equations to give

\[
-(\mathbf{b} + \mathbf{c}) = -\lambda (\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \lambda \mathbf{e}.
\]

This shows that \( \mathbf{EA} \parallel \mathbf{DB} \) and completes the proof.

Our second (pure) proof uses mainly area arguments and gets very quickly to the fifth parallel pair. The argument to bring in the Golden Ratio involves a generalization of the easier argument for the regular pentagon/pentagram case. Label the inner pentagon as \( PQRST \) as in the diagram. Since we have \( \mathbf{AB} \parallel \mathbf{EC} \), it follows that \( |\mathbf{ABE}| = |\mathbf{ABC}| \).

Following round the pentagon we obtain

\[
|\mathbf{ABE}| = |\mathbf{ABC}| = |\mathbf{BCD}| = |\mathbf{CDE}| = |\mathbf{DEA}|.
\]

This gives \( |\mathbf{EAB}| = |\mathbf{EAD}| \) and hence \( \mathbf{EA} \parallel \mathbf{DB} \).

Triangles \( \mathbf{AEQ} \) and \( \mathbf{BPC} \) are congruent (by \( \text{SAS} \)). So \( \mathbf{EQ} = \mathbf{PC} \) and hence \( |\mathbf{EDQ}| = |\mathbf{PDC}| \). Following around the pentagon we find

\[
|\mathbf{ABS}| = |\mathbf{BCT}| = |\mathbf{CDP}| = |\mathbf{DEQ}| = |\mathbf{EAR}|.
\]

Then \( |\mathbf{ERQ}| = |\mathbf{AEQ}| - |\mathbf{EAR}| = |\mathbf{BPC}| - |\mathbf{BCT}| = |\mathbf{CPT}| \), and following around again we find

\[
|\mathbf{ASR}| = |\mathbf{BTS}| = |\mathbf{CPT}| = |\mathbf{DQP}| = |\mathbf{ERQ}|.
\]

It follows that all the diagonals are divided in the same triple ratio, for example,

\[
|\mathbf{ER}| : |\mathbf{RS}| : |\mathbf{SB}| = |\mathbf{DP}| : |\mathbf{PT}| : |\mathbf{TB}| = x : 1 : x
\]

(say). Then \( |\mathbf{DB}|/|\mathbf{EA}| = |\mathbf{RB}|/|\mathbf{ER}| = (x + 1)/x \) and \( |\mathbf{DB}|/|\mathbf{EA}| = |\mathbf{DB}|/|\mathbf{DT}| = (2x + 1)/(x + 1) \). Hence \( x^2 = x + 1 \) and so \( x = G \). It follows that \( |\mathbf{DB}|/|\mathbf{EA}| = 1 + g = \mathcal{G} \).

**THEOREM 2.** Any two non-parallel vectors \( \mathbf{a} = \overrightarrow{AB}, \mathbf{b} = \overrightarrow{BC} \) uniquely determine a par-pentagon \( ABCDE \).
Proof. If there is such a par-pentagon $ABCDE$, then the other three sides must be determined by the vectors $c, d, e$ where

$$c = \overrightarrow{CD} = -a + (G - 1)b = -a + gb,$$
$$d = \overrightarrow{DE} = -ga - gb,$$
$$e = \overrightarrow{EA} = Ga - a - b = ga - b.$$

But it is routine to verify that these five vectors give five side-diagonal parallel pairs, and so the proof is complete.

EXERCISE 1. The points $D, E$ in Theorem 2 are uniquely determined by requiring that $AD \parallel BC$ and $BC = gAD$, and $EC \parallel AB$ and $|AB| = g|EC|$. Give a pure proof that $ABCDE$ is then a par-pentagon.

There is an affine transformation which maps $a, b$ to two adjacent edges of a regular pentagon. Since affine transformations preserve parallel lines, the image pentagon is a par-pentagon. But the regular pentagon is a par-pentagon. By the above uniqueness remark, the image pentagon is precisely the regular pentagon. Conversely, any affine image of a regular pentagon is a par-pentagon. We have thus proved:

**THEOREM 3.** A convex pentagon is a par-pentagon if and only if it is affinely equivalent to a regular pentagon.

In the figure given by a par-pentagon $ABCDE$, the five sides and the five diagonals form three angles at each vertex. Moving round the pentagon in the order $A, E, D, C, B$ we find by alternating angles between parallel lines that the five angle triples are of the form

$$\alpha, \beta, \gamma; \ \delta, \epsilon, \alpha; \ \beta, \gamma, \delta; \ \epsilon, \alpha, \beta; \ \gamma, \delta, \epsilon$$

where $\alpha + \beta + \gamma + \delta + \epsilon = \pi$. Conversely, any pentagon-pentagram with this angle pattern is a par-pentagon. In fact, it is easy to see that it is enough to have the pattern

$$\alpha_1, \beta, \gamma; \ \delta, \epsilon, \alpha_2; \ \beta, \gamma, \delta; \ \epsilon, \alpha_3, \beta; \ \gamma, \delta, \epsilon$$

since the angles of any triangle add to $\pi$.

Let $\Pi$ be a regular pentagon. It is well known that the intersections of the sides of the internal pentagram give another regular pentagon, similar to $\Pi$. Continuing the process gives a nested sequence of decreasing regular pentagons each similar to the previous one (always with the same similarity ratio $1 : G^2$). There is also a nested sequence of increasing regular pentagons. Extend each side of $\Pi$ and it is easy to verify that we obtain a regular pentagram whose vertices form a regular pentagon, similar to $\Pi$ and with the similarity ratio $G^2 : 1$. Since affine transformations preserve ratios on a line, it follows from Theorem 2 that each par-pentagon creates such nested sequences of similar par-pentagons.

By the center of gravity of a pentagon we mean the center of gravity of its five vertices. Recall that a median of a pentagon is the line from a vertex to the mid-point...
of the opposite side. For a regular pentagon Π the five medians are obviously concurrent at the center of the pentagon (which is in fact the center of gravity of the pentagon). An affine transformation maps the center of gravity of one pentagon to the center of gravity of the image pentagon. It thus follows from Theorem 2 that the five medians of a par-pentagon are concurrent at the center of gravity of its vertices. The converse is also true, giving another characterization of par-pentagons. In fact it is sufficient to have four medians concurrent at the center of gravity of the pentagon.

THEOREM 4. Let ABCDE be a pentagon in which the medians from A, B, C, D are concurrent at the center of gravity Γ of the five vertices of the pentagon. Then the median from E also passes through Γ, and ABCDE is a par-pentagon.

Proof. Let \( \vec{\Gamma A}, \vec{\Gamma B}, \vec{\Gamma C}, \vec{\Gamma D}, \vec{\Gamma E} \) represent the vectors \( a, b, c, d, e \) respectively. Thus \( a + b + c + d + e = 0 \). Let \( A', B', C', D', E' \) be the mid-points of the sides opposite \( A, B, C, D, E \) respectively. Thus \( \vec{\Gamma A'} = (c + d)/2 \), etc. For some positive reals \( \lambda, \mu, \nu, \rho \) we now have

\[
c + d = -\lambda a, \quad d + e = -\mu b, \quad e + a = -\nu c, \quad a + b = -\rho d.
\]

But these are precisely the vector equations in the proof of Theorem 1, and so we have \( \lambda = \mu = \nu = \rho = (\sqrt{5} + 1)/2 \) and \( b + c = -\lambda e \) so that \( E, \Gamma, E' \) are collinear. We have \( AB \parallel CE \) if and only if \( AB \parallel A'B' \) if and only if

\[
-a + b = k\left(\frac{1}{2}(c + d) - \frac{1}{2}(d + e)\right) = k\frac{1}{2}(c - e)
\]

and this is true from the first two vector equations. Similarly for the other four cases, so that \( ABCDE \) is a par-pentagon.

EXERCISE 2. In the notation of Theorem 3, verify that \( A'B'C'D'E' \) is also a par-pentagon. Give vector and pure proofs.

EXERCISE 3. Let \( \Pi = ABCDE \) be a convex pentagon. Show that the following are equivalent:

(i) \( \Pi \) is a par-pentagon;
(ii) The five medians of \( \Pi \) each bisect the area of \( \Pi \);
(iii) The medians from \( A, B, C, D \) each bisect the area of \( \Pi \).
EXERCISE 4. Give examples of pentagons which have exactly \( n \) side-diagonal parallel pairs, where \( n = 0, 1, 2, 3 \).

EXERCISE 5. Give an example of a pentagon for which the five medians are concurrent, but not at the center of gravity of the five vertices of the pentagon.

We have already seen that a convex par-pentagon \( ABCDE \) is determined by any pair of adjacent sides; for example, \( AB \) and \( BC \). It is also determined by one side and the opposite vertex; for example, \( AB \) and \( D \). To see the latter, observe that the directions of \( BC \) and \( AE \) are determined by \( AD \) and \( BD \) while their lengths are determined by \( |AD| \), \( |BD| \) and Golden Ratio. A consequence of this is that any three non-collinear points \( A \), \( B \) and \( C \) are three of the vertices of six distinct par-pentagons:

\[
CABXX', \; ABCYY'', \; BCAZZ',
\]

and

\[
BCPAP', \; CAQBQ', \; ABRCR'
\]

(say), the first three determined by two adjacent sides and the second three by one side and the opposite vertex.

EXERCISE 6. With the above notation show that
(a) the six par-pentagons with vertices including \( A, B, C \) are distinct,
(b) \( P, C, Q' \) are collinear and \( PQ \parallel AB \),
(c) \( R', C, X' \) are collinear and \( R'X' \parallel AR \),
(d) \( A, R, X \) are collinear,
(e) \( Y', P, C, Q', X \) are collinear,
(f) $XX'YY'ZZ'$ and $PR'QP'RQ'$ are par-hexagons (convex hexagons with opposite sides parallel, see [1]),

(g) the hexagons in (f) are tripar-hexagons (each (principal) diagonal and the two sides not adjacent to it are parallel),

(h) for the hexagon $XX'YY'ZZ'$, the lengths of each pair of opposite sides are in the (long : short) ratio $G^3 : 1$,

(i) for the hexagon $PR'QP'RQ'$, the lengths of each pair of opposite sides are in the (long : short) ratio $2G : 1$.

We leave the reader to further develop the multi-par-pentagon diagram and find more examples of collinear points and par-hexagons.

Thus far, our discussion of par-pentagons has been restricted to the case of convex pentagons. We have not given a formal definition of a **convex pentagon**, but the following is the most convenient for us. A pentagon (as a closed path in the plane) is **simple** if no two (non-adjacent) edges intersect; it is **convex** if it is simple and all internal angles are non-reflexive. It is essentially clear that this is equivalent to asking that the Jordan interior of a simple pentagon is a convex subset of the plane. For an arbitrary pentagon, the notion of the diagonal opposite a side is well-defined, and so we may define a **generalized par-pentagon** to be a pentagon for which each side is parallel to its opposite diagonal. For any convex par-pentagon, the associated pentagram (which is non-convex) is a generalized par-pentagon in that the side-diagonal pairs are the same for the pentagon and the pentagram. In fact we show that these are the only generalized par-pentagons. We say that a pentagon and its associated pentagram are dual to each other.

Let $\Pi = ABCDE$ be a generalized par-pentagon with $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EA}$ representing vectors $a, b, c, d, e$, respectively.

**LEMMA 1.** We have

$$a \times b = b \times c = c \times d = d \times e = e \times a.$$ 

**Proof.** Regard $a, b, c, d, e$, as (coplanar) vectors in 3-space. Since $a + b + c + d + e = 0$ we have $(a \times b) + (a \times c) + (a \times d) + (a \times e) = 0$, so that

$$(a \times b) + (a \times e) = -a \times (c + d) = 0 \quad \text{ (since } \Pi \text{ is a generalized par-pentagon)}.$$ 

Hence $a \times b = e \times a$ and, repeating this round $\Pi$, gives the required result.

**THEOREM 5.** Either $\Pi$ or its dual is convex.

**Proof.** Since $\Pi$ is a generalized par-pentagon, $AB\parallel EC$ and $BC\parallel AD$, and so there exist real numbers $\lambda$ and $\mu$ such that $\overrightarrow{AD} = \lambda\overrightarrow{BC}$ and $\overrightarrow{EC} = \mu\overrightarrow{AB}$. Then

$$c = (\lambda - 1)b - a, \quad d = (1 - \lambda)b + (1 - \mu)a, \quad e = (\mu - 1)a - b.$$ 

Hence

$$c \times d = (\lambda - 1)\mu(a \times b) \quad \text{and} \quad d \times e = \lambda(\mu - 1)(a \times b).$$
Since, by Lemma 1, \( c \times d = d \times e = a \times b \) it follows that

\[(\lambda - 1)\mu = \lambda(\mu - 1) = 1\]

from which we may deduce that \( \lambda = \mu = G \) or \( \lambda = \mu = -g \). Repeating round \( \Pi \) gives

\[
\overrightarrow{AD} = \rho \overrightarrow{BC}, \quad \overrightarrow{DB} = \rho \overrightarrow{EA}, \quad \overrightarrow{BE} = \rho \overrightarrow{CD}, \quad \overrightarrow{EC} = \rho \overrightarrow{AB}, \quad \overrightarrow{CA} = \rho \overrightarrow{DE},
\]

where either (i) \( \rho = G \) or (ii) \( \rho = -g \).

In case (i) \( \Pi \) is convex. To see that \( \Pi \) is simple, note that \( \overrightarrow{AD} = G \overrightarrow{BC} \). Then, since \( G > 0 \), it follows that \( AB \) and \( CD \) do not intersect. All other pairs of non-adjacent edges can be dealt with similarly. The convexity of \( \Pi \) follows from Lemma 1. If \( A \)-to-\( B \)-to-\( C \) involves a left turn (say) then so must \( B \)-to-\( C \)-to-\( D \), etc. Hence all internal angles are non-reflexive.

In case (ii) \( ACEBD \), the dual of \( \Pi \), is convex. Since \( \overrightarrow{EC} = (-g)\overrightarrow{AB} \) we have \( \overrightarrow{AB} = G \overrightarrow{CE} \) and it follows that \( ACEBD \) satisfies case (i).

REFERENCES