0 INTRODUCTION

These notes are intended as a supplement to a lecture course. It is assumed that the teacher will cover the material roughly in the order given here and certainly will use the ideas and proofs as given here. There are enough sample exercises given to allow for the construction of myriads of similar exercises, if needed. Thus, students taking this course do not need to buy an expensive textbook.

But this is not a traditional text. There are no pictures in the text; but the lectures should have lots of pictures on the blackboard as well as illustrative physical objects — and then you can draw your own pictures in the text. There is no color; but there may be lots of (fluorescent) colored chalk in the lectures. There are no displayed numbered theorems, there are no special messages in boxes, there are no end of chapter summaries; but there is a detailed list of contents to show how the plot unfolds. Nor are there any historical vignettes; the student who wishes these may care to read Huygens and Barrow, Newton and Hooke by Vladimir Arnold. I take the view that a first course on differential equations should be as short and as simple as it can be made. The majority of current texts unfortunately give the impression that this subject is a massive compendium of formulas and procedures. In fact, there are only a few ideas in this course and some of these come up again and again in slightly different guises. A few topics are taken for granted in the text, such as the “cover-up” rule for partial fractions, complex exponentials, etc. Of course these can be expounded in the lectures if the class is not familiar with them.

This is not a compendium. It is a book to be READ through — think of it as a novel whose plot is the unfolding of the first ideas in ordinary (as opposed to partial) differential equations. But it is not intended as fireside reading. Readers need paper and pencil to fill in some steps in the arguments that have been deliberately left to the reader. Of course these details are expounded in the lectures.

Please try to enjoy differential equations. They provide truly remarkable tools for the solution of problems in the physical and other sciences.

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Chapter 1  First order differential equations

Integrating factor methods, separation of variables, examples of non-uniqueness, exact equations, homogeneous equations of degree 0, existence and uniqueness of solutions for \( y' = f(x, y) \).

Chapter 2  Second order linear differential equations

The characteristic equation, general solutions, forcing terms and particular solutions, non-constant coefficients, Wronskians, variation of parameters, applications, higher order equations.

Chapter 3  The Laplace Transform

Definition, fundamental formulas (\( \mathcal{L}(y'(t)), \mathcal{L}(-ty(t)), \mathcal{L}(e^{at}y(t)) \)). Applications to differential equations. Methods for partial fractions. The convolution formula. The Heaviside function.

Chapter 4  Linear Systems

The elimination method. Solution of \( \mathbf{x}' = A\mathbf{x} \) by \( \mathbf{x} = e^{tA}\mathbf{x}(0) \). How to calculate \( e^{tA} \). Eigenvalues and eigenvectors. Invertible matrices. Methods for forcing terms. Use of Laplace Transforms. Non-constant coefficients. Wronskians. Proofs that \( e^{A}e^{B} = e^{A+B} \) when \( AB = BA \).

Chapter 5  Difference Equations

The characteristic equation, general solutions, forcing terms and particular solutions. Poincaré’s Theorem. Application to calculate \( e^{tA} \).
1 FIRST ORDER DIFFERENTIAL EQUATIONS

A differential equation is an equation that specifies a constraint on a function and one or more of its derivatives. Only one rate of change appears in a first order differential equation. For example, if a population \( p(t) \) has a rate of increase that is a fixed proportion of \( p(t)(N - p(t)) \) (where \( N \) is the limiting population over time), then it is governed by the first order differential equation

\[
\frac{dp}{dt} = cp(N - p).
\]

In this special case, the rate of change of the population depends only on the population and the rate of change does not vary with time. The most general first order differential equation is of the form

\[
\frac{dy}{dt} = f(t, y).
\]

At this level of generality there is no formula for a solution, but we can give formulas in several special cases (linear, separable, exact, and homogeneous). For the general case, we can use a geometrical method to get qualitative information about solutions, and for suitable functions \( f \) we can prove existence and uniqueness of solutions (even when we cannot produce nice formulas for the solutions). It is perhaps unfortunate that we begin with a plethora of special methods, but we shall see that most of the methods are essentially variants on a single theme. In the course of explaining these special methods we shall gradually introduce some fundamental notions and terminology that apply to differential equations in general.

We begin with the mathematician’s favorite case — the linear case. Here \( f(t, y) \) is a linear function of the variable \( y \), but the other terms are functions of \( t \). We shall study it in the form

\[
y' + p(t)y = q(t).
\]

We are already familiar with the special case \( y' = y \). For many of us, the exponential function is defined via the differential equation \( y' = y \) with initial value \( y(0) = 1 \). We begin by assuming that such a function exists and go on, by elementary calculus, to prove that only one such function exists and to derive its properties. We recall the argument now, since it presents much
of the theory of differential equations in microcosm. Choose any \( c \in \mathbb{R} \) and define the function \( u \) by

\[
u(t) = y(t)y(c - t).
\]

A simple computation shows that \( u'(t) = 0 \) for all \( t \) and so \( u \) is constant,

\[
u(t) = u(0) = y(0)y(c) = y(c).
\]

Change the notation by writing \( a = t, c = a + b \) and we get the “law of indices”

\[
y(a + b) = y(a)y(b)
\]

for all real \( a, b \). Put \( b = -a \) to get \( y(a)y(-a) = 1 \). In particular, \( y(a) \) can never be zero. Already we have derived a qualitative property of the solution that was not at all evident \textit{a priori}. Since \( y(t) = y(t/2)y(t/2) \) we now find that \( y(t) > 0 \) for all \( t \), yet another qualitative property. Since \( y' = y > 0 \), we deduce that \( y \) is strictly increasing. Since \( y'' = y' > 0 \), we also deduce that \( y \) is convex up. Now let us see that there is unique solution. Suppose that \( z(t) \) is any function with \( z' = z \) and \( z(0) = 1 \). Since \( y \) is always positive, we may define a function \( u \) by \( u(t) = z(t)/y(t) \) for all \( t \). An easy computation shows that \( u'(t) = 0 \) for all \( t \), and so \( u \) is a constant function with value \( u(0) = z(0)/y(0) = 1 \). Hence \( z(t) = y(t) \), as claimed.

Now we are ready to turn this argument to the more general case

\[
y' + p(t)y = 0, \quad y(a) = y_0.
\]

How can we get an associated function \( u(t) \) so that \( u'(t) = 0 \) for all \( t \)? Just take \( u(t) = e^{P(t)}y(t) \) where \( P'(t) = p(t) \). Then we get

\[
u'(t) = e^{P(t)}[y'(t) + p(t)y(t)] = 0
\]

for all \( t \). Thus \( u \) is constant, say \( c \), and we get \( y(t) = ce^{-P(t)} \). We call this formula the general solution of this differential equation, because these are the only formulas which satisfy \( y' + p(t)y = 0 \). As \( c \) varies, we get a family of curves no two of which ever meet each other. It is easy to see that this family of curves fills the whole plane; we call this a foliation of the plane. As soon as we specify one point on a solution curve, we uniquely determine the only solution curve through that point. In the present situation, put \( t = a \), and then we uniquely determine the value of \( c \) in terms of \( y_0 \). Of course
there are infinitely many different functions $P(t)$ with $P′(t) = p(t)$, but they differ only by constants and all produce the same foliation of the plane into solution curves. It is convenient to choose the integral formula for $P(t)$ as

$$P(t) = \int_a^t p(\tau) \, d\tau.$$  

Then we have $P(a) = 0$ and the corresponding value of $c$ is just $y_0$. Notice that we have proved both existence and uniqueness. As a very simple example, take $y′ - 2ty = 0$ with $y(0) = 1$, and we easily get $y(t) = e^{t^2}$. The function $e^{P(t)}$ is called the integrating factor for the differential equation.

Now we can take one further step by generalizing to the case

$$y′ + p(t)y = q(t), \quad y(a) = y_0.$$  

When we multiply by the integrating factor $e^{P(t)}$ our differential equation becomes

$$\frac{d}{dt}[e^{P(t)}y(t)] = e^{P(t)}q(t)$$

and our problem is reduced to integrating the function on the right side of the equation. This will introduce an arbitrary constant (thereby giving the general solution), but when we put $t = a$ we shall find the value of the constant. Again we have existence and uniqueness. In practice we have to be able to do two integrals. For example, consider the differential equation

$$y′(t) - 2ty(t) = t^3, \quad y(0) = 0.$$  

We can take the integrating factor as $e^{-t^2}$ and so we get

$$\frac{d}{dt}[e^{-t^2}y(t)] = t^3e^{-t^2}.$$  

Integrate by substitution and parts to get

$$e^{-t^2}y(t) = -\frac{1}{2}t^2e^{-t^2} - \frac{1}{2}e^{-t^2} + c.$$  

Put $t = 0$ to get $c = 1/2$ and finally cross-multiply to get

$$y = \frac{1}{2}e^{t^2} - \frac{1}{2} - \frac{1}{2}t^2.$$
Notice how much easier the general theory is than the actual example!

On the face of it, there seems no hope of extending the integrating factor idea to differential equations that are not linear in $y$. But note that a key idea was to reduce our problem to performing integrations. To link in with ideas from Calculus III we are now going to change the variable names to $x$ and $y$. Suppose that $f(x, y) = p(x)q(y)$ (recall how nice this was for double integrals!). Our differential equation now becomes $y' = p(x)q(y)$, or, more conveniently,

$$\frac{dy}{q(y)} = p(x)dx.$$  

Unsurprisingly, this method is called separation of variables. We put integral signs on each side and integrate away! [If you have any doubts about the validity of this approach, you can always check that the final formula does actually solve the differential equation.] Let’s illustrate by solving the population equation $y' = cy(N - y)$ with initial value $y(0) = N/2$. We separate variables to get

$$\frac{dy}{y(N - y)} = cdx.$$

Now we use partial fractions to get

$$\left( \frac{1}{y} + \frac{1}{N - y} \right) dy = cN dx$$

and hence

$$\log \left( \frac{y}{N - y} \right) = cNx + k.$$ 

Put $x = 0$ and solve to get $k = 0$. Exponentiate and rearrange to get

$$y = \frac{e^{cNx}}{1 + e^{cNx}N}.$$

This is called a logistic curve. Of course we can obtain its properties from the actual formula, but in fact it is easier to do so by working with the differential equation.

To simplify the discussion, let’s take $c = N = 1$. So we have $y' = y(1 - y)$. Note first that $y = 0$ and $y = 1$ give constant solutions to the differential equation. For $0 < y < 1$ we have $y' > 0$. This means that any solutions confined between 0 and 1 must be increasing functions. Similarly,
any solutions with $y > 1$ (or with $y < 0$) must be decreasing functions. What about convexity and points of inflection? Since $y' = y - y^2$ we get $y'' = (1 - 2y)y' = (1 - 2y)y(1 - y)$. Thus $y$ is convex up when $0 < y < 1/2$, is convex down when $1/2 < y < 1$ and has a point of inflection when $y = 1/2$. Also, $y$ is convex up when $y > 1$ and convex down when $y < 0$. Notice that we have derived all this information without having any idea what the formula for $y$ looks like. By slightly more sophisticated calculus reasoning we can discover even more, if we assume that there is a unique solution $y$, for all $x$, passing through any given point in the plane. For $1/2 < y < 1$, the formula for $y''$ shows that $y'$ is positive and decreasing and so has a limit at $+\infty$. This limit cannot be positive. Indeed, suppose towards a contradiction that the limit is $\delta > 0$. For any $T > a$, the Mean Value Theorem gives

$$y(T) - y(a) = y'(\xi)(T - a) \geq \delta(T - a).$$

But this makes the solution curve cross $y = 1$ and so contradicts the uniqueness of the solution through that point (namely, $y = 1$). So the limit of $y'$ must be 0. Since $y' = y(1 - y)$ and $1/2 < y < 1$, it follows that the limit of $y$ at $+\infty$ must be 1. Similarly the limit of $y$ at $-\infty$ has to be 0. We have thus been able to derive the qualitative behavior of the solution curve without any information about its formula. The message is thus: differential equations encode lots of information about their solutions and that information is available without knowing a formula for the solution. This last analysis can be carried out much more generally for the differential equation $y' = q(y)$. We get $y'' = q'(y)y' = q'(y)q(y)$ and so we can carry out the above analysis provided we know where $q, q'$ are zero, positive and negative.

The above argument to establish the asymptotic behavior was a little sophisticated (we had to use an existence and uniqueness theorem). Here is another technique, which uses only elementary calculus, and even gives quantitative information. Let’s consider the solution of $y' = ye^{-y}$, just for $y > 0$. We can separate the variables, but we can’t carry out the integration with respect to $y$. We begin with the usual geometrical analysis. Since $y' > 0$, each solution $y$ is increasing. We get $y'' = (e^{-y} - ye^{-y})y' = y(1 - y)e^{-2y}$ and hence $y$ is convex up for $0 < y < 1$ and convex down for $y > 1$, with $y'$ greatest at $y = 1$. So we know the shape of the solution curves. But
what happens at infinity? Is \( y \) bounded above as \( t \to \infty \)? Or does \( y \to \infty \) and correspondingly \( y' \to 0 \), as happens, say, with the function \( y = t^{1/2} \)? The idea is to look at the key component of the order of magnitude of \( y' \) as \( t \to \infty \). It looks like this is given here by \( e^{-y} \). Let’s try to compare our solutions with those of the differential equation \( z' = e^{-z} \). We can solve this by separation of variables to get \( z = \log(t + c) \). We are interested only in what happens when \( y > 1 \) and then we have

\[
y' = ye^{-y} > e^{-y} = z'.
\]

For solutions with \( y(a) = z(a) \) it follows by elementary calculus that \( y(t) > z(t) \) when \( t > a \). Thus \( y \) grows faster than a logarithmic curve (and so does indeed go to \( \infty \) as \( t \to \infty \)). Can we also get an upper estimate for how fast \( y \) grows? Let \( 0 < k < 1 \). By elementary calculus we have \( kye^{-ky} \leq 1/e \) and it follows that

\[
ye^{-y} \leq (1/ek)e^{-(1-k)y}.
\]

We now compare \( y \) with the solutions of the differential equation \( w' = (1/ek)e^{-(1-k)w} \) and we find that \( y \) is bounded above by a function of the form

\[
w = \frac{1}{1-k} \log\left(\frac{1-k}{ek}t + (1-k)c\right).
\]

Hence the growth of our solutions \( y \) is indeed logarithmic. We can do a similar analysis as \( t \to -\infty \). Here the key component of the order of magnitude of \( y' \) is \( y \), and the solutions \( y \) behave like negative exponentials.

Thus far, “everything in the garden has been rosy”. It is time to look at a troubling example. Consider the differential equation \( y' = 3y^{2/3} \) with initial value \( y(0) = 0 \). We can separate the variables to give

\[
\frac{dy}{3y^{2/3}} = dx
\]

and integrate to give \( y^{1/3} = x + c \). Put \( x = 0 \) to get \( c = 0 \) and hence \( y = x^3 \). What’s wrong with that? Nothing, on the face of it; but notice that we get another solution by taking \( y = 0 \). So the solution is not unique. But it gets worse. Take \( a > 0 \) and define \( y \) by \( y(x) = 0 \) for \( x < a \) and \( y(x) = (x-a)^3 \) for \( x \geq a \). We readily verify that this also gives a solution. Now we have infinitely many solutions with the same value at \( x = 0 \)! We can double the infinity of solutions by taking \( b < 0 \) and then redefining \( y \) for \( x < b \) by
$y(x) = (x-b)^3$. We shall see later why our general theorem on existence and uniqueness does not apply to this example. But why did the separation of variables method miss this difficulty? The integral with respect to $y$ had a singularity at $y = 0$ which we conveniently ignored. When $y(x) = 0$ for an interval of $x$ values, we simply cannot perform the integration. We avoided this difficulty in the above population example by restricting ourselves to the condition $0 < y < 1$. Does the differential equation $y' = y(1-y)$ have unique solution with $y(0) = 0$. Yes, it has unique solution $y = 0$, but that is not quite obvious — can you prove it?

What hope do we have in general when the variables $x, y$ do not separate? There is one other easy case which comes from a fundamental theme in vector calculus, namely gradient vector fields. Let’s change the notation again and consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0.$$  

When the vector field $< M, N >$ is a gradient vector field, $\nabla f$, then we get

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy = 0$$

and so the differential equation is solved by the formula $f(x,y) = c$. [We say that such an equation is exact.] As an illustration, note that the equation

$$(x^2 + y)dx + (y^2 + x)dy = 0$$

is solved by $x^3 + y^3 + 3xy = c$. When the functions $M, N$ have no singularities, the vector field $< M, N >$ is a gradient field provided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. In practice, it is not worth bothering to check this - we just seek the function $f(x,y)$ (as in Calculus III). In many examples, the functions $M, N$ are polynomials and we can usually see the answer by using the formulas

$$d(x^ay^b) = ax^{a-1}y^b dx + bx^ay^{b-1} dy.$$  

When we allow $a, b$ to be positive or negative we can also accommodate rational functions.

There are lots of other special cases where solutions can be found. We’ll do just one more. Recall from Calculus III that $f(x,y)$ is homogeneous of
degree $k$ if $f(tx, ty) = t^k f(x, y)$. We can handle rational functions that are homogeneous of degree 0 (but usually it is very hard work). We illustrate by one example. Consider $(x^2 + y^2)y' = xy$ with $y(1) = 1$. Write $y = xv$ so that $y' = xv' + v$ and the differential equation becomes

$$xv' + v = \frac{v}{1 + v^2}.$$ 

Hence

$$xv' = -\frac{v^3}{1 + v^2}$$

and we are reduced to a separable case! We integrate to get

$$-\frac{1}{2v^2} + \log v = -\log x + c.$$ 

Put $x = 1$ to give $v = 1$ and so $c = -1/2$. Now rearrange and simplify to end up with solution curve $x^2 - y^2 = 2y^2 \log y$. Note that we could obtain $x$ as a (horrid) function of $y$. The above is typical of the contorted formulas that arise from homogeneous differential equations.

We have been able to solve many first order differential equations and these equations give reasonable models for many applied problems (see the exercises); but if we choose a function $f(x, y)$ at random it will not fit into any of the above categories. A very simple such example is given by $y' = x^2 + y^2$. It is not linear, nor homogeneous of degree 0, the variables do not separate and it does not come from a gradient vector field. So what can we do? We can think about the differential equation geometrically. The equation says that the slope of the tangent line at the point $(x, y)$ has the value $x^2 + y^2$. This gives a vector field in the plane, but we can’t see all of it at once. We choose a grid of points in the plane and at each point $(x, y)$ of the grid we draw a little vector with slope $x^2 + y^2$. By eyeball, we try to see curves in the plane that have these tangent vectors. This is surprisingly easy if we have a large grid of points — but it takes forever to draw a careful picture with a huge number of vectors. It is natural to opt for a uniform grid of points in the plane, but in practice it is usually better to draw the contour lines for $f(x, y)$. In our case, at every point of the circle $x^2 + y^2 = c^2$ the tangent vector has slope $c^2$. The vector field picture gives a rough idea of what the solutions look like, but it is not good at predicting what happens in delicate situations. Try drawing vector fields for the two equations $y' = y(1 - y)$ and
\[ y' = 3y^{2/3} \] and see if you can distinguish the behavior of the solutions! Even computer software fails here (though it does save a lot of tedious drawing!)

Finally we turn to the question of existence and uniqueness of solutions for \( y' = f(x, y) \). We have already seen that uniqueness can fail. Our special methods guaranteed existence and usually the solution was defined for all values of \( x \), but we did have cases where the formulas for the solutions implied that the solution was valid, say for only \( x > 0 \). To see what to expect in general, consider two simple examples. For \( y' = y^2 \) with \( y(0) = 1 \), we easily find the solution to be \( y(x) = 1/(1 - x) \) and clearly the solution “ends” at \( x = 1 \). For \( y' = 1 + y^2 \) with \( y(0) = 0 \), we easily find the solution to be \( y(x) = \tan(x) \) and clearly the solution “ends” at both \( x = \pi/2 \) and \( x = -\pi/2 \). These simple examples show that the general story has to be complicated and we cannot expect to predict where the solution will end just by looking at the differential equation!

We begin with the problem of existence for the differential equation \( y' = f(x, y) \) with initial value \( y(x_0) = y_0 \). The function \( f \) has to be reasonably nice — we shall make clearer how nice it has to be as we proceed. It is a well known fact of calculus that, for theoretical purposes, integration is a much nicer operator than differentiation. So we rewrite our differential equation as an integral equation:

\[ y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t)) \, dt. \]

The equivalence of the two equations is an immediate consequence of the fundamental theorem of calculus. To keep the formulas a little simpler we are going to suppose that \( x_0 = 0 \). Of course, \( f \) has to be nice enough for the fundamental theorem to work; we may as well suppose that \( f \) is continuous on some rectangle containing \((0, y_0)\). As a first guess for a solution we might as well try \( y_1(x) = y_0 \). Let’s define \( y_2(x) \) by

\[ y_2(x) = y_0 + \int_{0}^{x} f(t, y_1(t)) \, dt. \]

If \( y_2 = y_1 \) then we have found a solution; if not, we can try again by defining \( y_3(x) \) by

\[ y_3(x) = y_0 + \int_{0}^{x} f(t, y_2(t)) \, dt. \]
If \( y_3 = y_2 \) then we have a solution; if not we keep on trying by this algorithm and we construct a sequence of functions \( y_n(x) \). At the \( n \)-th stage we have

\[
y_{n+1}(x) = y_0 + \int_0^x f(t, y_n(t)) dt.
\]

Now we should like to let \( n \) go to \( \infty \) and hope that \( y_n(x) \) converges nicely to a limit function \( y_\infty(x) \) which will then satisfy the equation

\[
y_\infty(x) = y_0 + \int_0^x f(t, y_\infty(t)) dt
\]

in other words \( y_\infty \) will be a solution to the differential equation.

How are we to prove all this? What properties must \( f \) have for all this to work? It turns out to be helpful to change the sequence \( y_n \) into a series via

\[
y_n = y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}).
\]

So we need to estimate the size of successive differences. We easily get

\[
y_{n+1}(x) - y_n(x) = \int_0^x [f(t, y_n(t)) - f(t, y_{n-1})] dt.
\]

Suppose that

\[
|f(t, u) - f(t, v)| \leq L |u - v|
\]

whenever \( 0 \leq t \leq a \) with \( u, v \) arbitrary. Write \( ||g|| \) for the largest value of \( |g(t)| \) for \( 0 \leq t \leq a \). We now get

\[
||y_{n+1} - y_n|| \leq \int_0^x L ||y_n - y_{n-1}|| dt = Lx ||y_n - y_{n-1}||.
\]

Now use this estimate to deduce that

\[
||y_{n+1} - y_n|| \leq (1/2) L^2 x^2 ||y_{n-1} - y_{n-2}||.
\]

Continue in this fashion to end up with

\[
||y_{n+1} - y_n|| \leq ||y_2 - y_1||(Lx)^n / n!.
\]

It is now clear that our series converges absolutely by comparison with the exponential series, and so has limit \( y_\infty(x) \). Since

\[
|f(t, y_n(t)) - f(t, y_\infty(t))| \leq ||y_n - y_\infty||
\]
it is straightforward to verify that \( y_\infty \) gives a solution to the differential equation.

Suppose that \( z_\infty \) is another solution. But then we have

\[
y_\infty(x) - z_\infty(x) = \int_0^x [f(t, y_\infty(t)) - f(t, z_\infty(t))]dt
\]

and hence \( \|y_\infty - z_\infty\| \leq Lx\|y_\infty - z_\infty\| \), and this is a contradiction unless \( \|y_\infty - z_\infty\| = 0 \), that is, \( y_\infty = z_\infty \).

This is a substantial piece of mathematical reasoning, but unfortunately it is not quite good enough for most applications. The key requirement is satisfied if \( \partial f/\partial y \) is bounded in a vertical strip. But this does not hold true for the case \( f(y) = y^2 \) which we discussed above. Suppose that \( \partial f/\partial y \) is bounded on a rectangle containing \((0, y_0)\). Then we can modify the above argument (details suppressed!) to show that a unique solution exists on some interval \([0, a]\) (perhaps only a very small interval). This theorem can be applied to the two examples we considered above. But still we have no effective method to determine the largest interval on which a solution exists. On the other hand we can prove (again details suppressed) that if \( \partial f/\partial y \) is bounded on a rectangle containing two solution curves through the point \((0, y_0)\) then the two solutions coincide. This enables one to show, for example, that for any initial value \( y(0) \) there is a unique solution to the differential equation \( y' = y(1 - y) \).

The inequality \( |f(t, u) - f(t, v)| \leq L|u - v| \) is called the Lipschitz condition. It is a weaker requirement than asking that \( \partial f/\partial y \) be bounded, and it enables us to handle problems in economics where sudden jumps occur in the data and the partial derivative does not exist everywhere (see advanced texts on differential equations).
1. Find the general solution of
   (i) \( y' + \frac{2}{t}y = 1 + 2t^2 \)
   (ii) \( y' - \frac{1}{t}y = e^{-t^2} \)
   (iii) \( y' + (\cot t)y = t \)

2. Solve
   (i) \( y' + ty = e^{-t}; \quad y(0) = 1 \)
   (ii) \( y' + e^ty = e^t; \quad y(0) = 0 \)
   (iii) \( y' - (\tan t)y = t; \quad y(\pi/2) = 0 \)

3. Find the general solution of
   (i) \( (1 - 2y)y' = te^{-t} \)
   (ii) \( ty^2y' = \log t \)
   (iii) \( y' = y(1 - y^2) \)

4. Solve
   (i) \( (y + y^2)y' = \sin t; \quad y(\pi) = 1 \)
   (ii) \( (1 + t^2)yy' = t; \quad y(0) = 1 \)
   (iii) \( (1 + t^2) \sin yy' = 1; \quad y(\pi/2) = 0 \)

5. Discuss the solutions of
   (i) \( y' = ye^{y^2} \)
   (ii) \( y' = 2^2 + y^2 \)
   (iii) \( y' = \sqrt{(1 + y^2)} \)
   (iv) \( y' = y\sqrt{(1 - y^2)} \)

6. Supply the details to establish the behavior of the solutions of \( y' = ye^{-y} \) near \( y = 0 \).

7. Find the general solution of
   (i) \( (y^2 + x)\frac{dy}{dx} + x^3 + y = 0 \)
   (ii) \( (xy - \frac{1}{y})\frac{dy}{dx} + y^2 = 0 \)
   (iii) \( (\sin x + x^2 \sin y)\frac{dy}{dx} = 2x \cos y - y \cos x \)

8. Solve
   (i) \( (x + y)y' = (x - y); \quad y(1) = 0 \)
   (ii) \( \frac{dy}{dx} = \frac{2xy}{y^2 - x^2}; \quad y(1) = 1 \)
2 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Lots of applied problems (vibrating springs, electrical meters, LRC circuits, etc) can be reasonably described by a differential equation of the form

\[ y'' + \alpha y' + \beta y = 0 \]

where \( \alpha, \beta \) are constants; or more generally by

\[ y'' + \alpha y' + \beta y = g(t) \]

where \( g(t) \) is sometimes called an external forcing term. We can easily solve these equations by reducing the problem to the solution of two first order linear differential equations.

We illustrate the method with a simple example and then treat the general case. Suppose we wish to solve \( y'' - 2y' = 0 \). Let \( w = y' \) and we have to solve \( w' - 2w = 0 \). We know this has general solution \( w = Ae^{2t} \). Now we have to solve \( y' = Ae^{2t} \). This has general solution \( y = (A/2)e^{2t} + B \), where \( A, B \) are any constants. Now we consider the general case

\[ y'' + \alpha y' + \beta y = 0. \]

Let \( D = d/dt \) and let \( I \) be the identity operator (that is, \( Iy = y \)). We can rewrite the equation as

\[ (D^2 + \alpha D + \beta I)y = 0. \]

Since we have a quadratic formula we are tempted to factorize it. We know that we can factorize

\[ z^2 + \alpha z + \beta = (z - \lambda)(z - \mu) \]

where \( \lambda, \mu \) may have to be complex numbers. We then get the same factorization of our differential operator

\[ D^2 + \alpha D + \beta I = (D - \lambda I)(D - \mu I). \]

Just make this operator act on \( y \) to see that this works. (Caution: it is critical here that \( \alpha, \beta \) are constants — when they are functions of \( t \) we have to change the formula!) Now we have to solve the differential equation

\[ (D - \lambda I)(D - \mu I)y = 0. \]
We just mimic the special example above. Let \( w = (D - \mu I)y \), and we have to solve \((D - \lambda I)w = 0\), or \( w' - \lambda w = 0 \). We know this has general solution \( w = Ae^{\lambda t} \). Now we have to solve \((D - \mu I)y = Ae^{\lambda t}\), or \( y' - \mu y = Ae^{\lambda t} \). This equation has integrating factor \( e^{-\mu t} \) and so we get
\[
D(e^{-\mu t}y) = Ae^{(\lambda-\mu)t}.
\]

Case 1: \( \lambda \neq \mu \). We integrate to get
\[
e^{-\mu t}y = \frac{A}{\lambda - \mu} e^{(\lambda-\mu)t} + B
\]
and hence
\[
y = C_1 e^{\lambda t} + C_2 e^{\mu t}
\]
where \( C_1, C_2 \) are arbitrary constants.

Case 2: \( \lambda = \mu \). We integrate to get
\[
e^{-\mu t}y = At + B
\]
and hence
\[
y = (C_1 t + C_2) e^{\mu t}
\]
where \( C_1, C_2 \) are arbitrary constants.

This gives us a complete algorithm to solve our differential equation. We just solve the characteristic equation \( z^2 + \alpha z + \beta = 0 \) to get \( z = \lambda, \mu \) and then write down the appropriate general solution, as above. We pause to ask: what happens if the roots are complex? Since our \( \alpha, \beta \) are always real, if \( \lambda \) is complex, say \( \lambda = a + ib \), then \( \mu \) is just the complex conjugate, \( \mu = a - ib \). Recall that
\[
e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt).
\]
It follows that the general solution in this case may be written in the form
\[
y = e^{at}(A_1 \cos bt + A_2 \sin bt)
\]
where \( A_1, A_2 \) are arbitrary real constants. (Work out for yourself the relation between the complex constants \( C_1, C_2 \) and the real constants \( A_1, A_2 \).)

Notice that these general solutions all involve only well known functions (we need more complicated differential equations to lead us to new special
functions). In each case, our equation has a (double) infinity of answers. What extra information do we need to specify one definite formula for the solution? If we know the initial values \( y(p), y'(p) \) for just one value of \( p \), then the arbitrary constants become completely determined; in other words, the solution to our differential equation is uniquely determined by the initial values \( y(p), y'(p) \). Let’s prove this in the easy case when \( p = 0 \); you can do the general case (with a hint). We have to treat the two cases separately.

**Case 1**: \( \lambda \neq \mu \). We have \( y = C_1 e^{\lambda t} + C_2 e^{\mu t} \) and hence \( y' = \lambda C_1 e^{\lambda t} + \mu C_2 e^{\mu t} \).

Put \( t = 0 \) and we get the equations

\[
C_1 + C_2 = y(0), \quad \lambda C_1 + \mu C_2 = y'(0).
\]

Since \( \lambda \neq \mu \), these linear equations have unique solution for \( C_1, C_2 \).

**Case 2**: \( \lambda = \mu \). We have \( y = (C_1 t + C_2) e^{\mu t} \) and hence

\[
y' = C_1 e^{\mu t} + \mu (C_1 t + C_2) e^{\mu t}.
\]

Put \( t = 0 \) and we get the equations

\[
C_2 = y(0), \quad C_1 + \mu C_2 = y'(0).
\]

Clearly these linear equations have unique solution for \( C_1, C_2 \).

To deal easily with the general case, and to solve actual numerical examples, note that the general solutions may just as well be written in the forms

\[
y = B_1 e^{\lambda (t-p)} + B_2 e^{\mu (t-p)}, \quad y = (B_1 t + B_2) e^{\mu (t-p)}.
\]

Suppose that we know the values of \( y \) for two values, say we know \( y(p) \) and \( y(q) \). Most often, this will lead to a unique solution, but there are troublesome cases where uniqueness, and even existence, fails. There is no solution of \( y'' + y = 0 \) with \( y(0) = 0 \) and \( y(2\pi) = 1 \), since all solutions are periodic with period \( 2\pi \). Moreover, there is more than one solution with \( y(0) = y(2\pi) = 0 \); we may take \( y = 0 \) or \( y = \sin t \). Similar remarks apply when we know \( y'(p) \) and \( y'(q) \). What happens if we know \( y(p) \) and \( y'(q) \)? Over to you!

The qualitative nature of the solution is almost completely characterized by the characteristic equation. If the roots are, say, 1 and 2, then the general
solution is \( Ae^t + Be^{2t} \) and so \(|y|\) goes off to infinity unless both \( A = 0 \) and \( B = 0 \). In practice we shall not have these exact values for \( A \) and \( B \) and so “all” solutions will go to infinity. If the roots are, say, -1, -2, then the general solution is \( Ae^{-t} + Be^{-2t} \) and so every solution dies out to 0. When we get a mixture of signs, say, -1, 2, then the general solution is \( Ae^{-t} + Be^{2t} \) and so \(|y|\) goes to infinity unless \( B = 0 \), which we cannot achieve exactly in practice. With repeated roots, say, -1, -1, then the general solution is \((At + B)e^{-t}\) and again all solutions die out to 0. When the repeated root is positive, then “all” solutions go to infinity. When the roots are complex, say, \( 1 \pm 2i \), then the general solution is \( e^{t}(A \cos 2t + B \sin 2t) \) and we get wildly increasing oscillations. With roots \( -1 \pm 2i \), we get general solution \( e^{-t}(A \cos 2t + B \sin 2t) \) and we get a damped oscillation rapidly dying out to 0. For behavior at infinity, the critical question always is: what are the real parts of the roots?

Now let’s see what happens when we have a forcing term \( g(t) \) on the right side of the equation. To simplify notation, write \( L = D^2 + \alpha D + \beta I \).

Note that \( L \) is linear in the sense that \( L(ku) = kLu, L(u + v) = Lu + Lv, \) where \( u, v \) are functions and \( k \) is a constant. We want to solve \( Ly = g \).

Suppose that \( y_0 \) is a particular solution and that \( z \) solves \( Lz = 0 \). Then \( L(y_0 + z) = Ly_0 + Lz = g + 0 = g \), and so \( y_0 + z \) solves our equation. Conversely, let \( y \) be any solution of \( Ly = g \). Define \( z = y - y_0 \). Then \( Lz = Ly - Ly_0 = g - g = 0 \), and \( y = y_0 + z \). So we have proved that the general solution of \( Ly = g \) is given by \( y_0 + z \) where \( y_0 \) is one particular solution of \( Ly = g \) and \( z \) is the general solution of \( Lz = 0 \).

How can we find a particular solution? There is a fail-safe method when \( g(t) \) is of the form \( e^{\gamma t}p(t) \) where \( p(t) \) is a polynomial function. Consider first the case when we just have a polynomial. We illustrate with a specific example — clearly we can handle any polynomial, just with more computational effort. We want to find a solution of

\[
(D^2 - 3D + 2I)y = t^2 - t.
\]

Then \( y \) must also satisfy the differentiated equation

\[
(D^3 - 3D^2 + 2D)y = 2t - 1.
\]

Is there an obvious solution to this? Well, not quite, but let’s differentiate once more to get

\[
(D^4 - 3D^3 + 2D^2)y = 2.
\]
Is there an obvious solution to this? Yes, indeed! Just take $2D^2y = 2$ and this forces $D^3y = D^4y = 0$. Substitute these values back into the previous equation to get

$$0 - 3 + 2Dy = 2t - 1, \quad 2Dy = 2t + 2, \quad Dy = t + 1.$$ 

Finally, substitute back into the original equation to get

$$1 - 3(t + 1) + 2y = t^2 - t, \quad 2y = t^2 + 2t + 2, \quad y = (1/2)t^2 + t + 1.$$ 

It should now be clear that this “differentiate-and-back-substitute” technique will work for any polynomial. We just keep differentiating until an answer is obvious and then we back substitute.

What happens with a situation like $(D^2 - 3D)y = t$? We differentiate to get $(D^3 - 3D^2)y = 1$, and so we take $D^2y = -(1/3)$. Now back substitute to get $-3Dy = t + 1/3$. Finally we integrate to get one solution as $y = -t^2/6 - t/9$.

Now suppose that $g(t) = e^{\gamma t}p(t)$. We reduce the problem of finding a particular solution to the polynomial case by a simple idea. Write $y = e^{\gamma t}w$.

Check for yourself that

$$D(e^{\gamma t}w) = e^{\gamma t}(D + \gamma I)w, \quad D^2(e^{\gamma t}w) = e^{\gamma t}(D + \gamma I)^2w.$$ 

Hence the equation

$$(D^2 + \alpha D + \beta I)y = e^{\gamma t}p(t)$$

is reduced to the equation

$$((D + \gamma I)^2 + \alpha(D + \gamma I) + \beta I)w = p(t).$$

But we know how to get a particular solution $w$ of this equation, and hence we get a particular solution $y = e^{\gamma t}w$ to our original equation.

More generally, suppose that $g(t)$ is the sum of two terms of the above type, say $g(t) = g_1(t) + g_2(t)$. To solve $Ly = g$, we just solve $Ly_1 = g_1$ and $Ly_2 = g_2$, and then $y = y_1 + y_2$ gives a particular solution to $Ly = g$. Since $\cos 2t = (1/2)e^{2it} + (1/2)e^{-2it}$, we can cope with cosine and sine terms on the right side. In practice, it is usually quicker to replace $\cos 2t$ by $e^{2it}$, solve the complex differential equation and then take the real part. We illustrate with an example. Find a particular solution of

$$(D^2 + D + I)y = t \cos t.$$
We begin by finding a particular solution of
\[(D^2 + D + I)z = te^{it}\.\]
As a first step we write \(z = e^{it}w\) and now we have to solve
\[((D + iI)^2 + (D + iI) + I)w = t\]
that is,
\[(D^2 + (1 + 2i)D + iI)w = t\.
We differentiate to get
\[(D^3 + (1 + 2i)D^2 + iD)w = 1\.
So we take \(iDw = 1\), \(D^2w = 0\). Now back substitute to get
\[0 + (1 + 2i)(-i) + iw = t, \quad w = -it + 1 + 2i.\]
This gives us the solution
\[z = e^{it}w = (\cos t + i \sin t)(-it + 1 + 2i)\.
Our sought for solution is just the real part of \(z\), that is \(y = \cos t + (t-2) \sin t\). Notice that we have to do some arithmetic with complex numbers.

The above algorithm never fails. There are shortcuts in special cases, which you may acquire for yourself.

What do we do if \(g(t)\) cannot be written as a sum of the above types of function? There is another general algorithm, but in practice we cannot carry it out since it usually involves integrals that we cannot do explicitly.

Recall that we factorized \(D^2 + \alpha D + \beta I\) as \((D - \lambda I)(D - \mu I)\). So we have to solve the differential equation
\[(D - \lambda I)(D - \mu I)y = g\.
We use the same method that we started with! Write \(w = (D - \mu I)y\). So we begin by solving
\[(D - \lambda I)w = g, \quad w' - \lambda w = g.\]
By the integrating factor method, a solution is given by
\[w(t) = e^{\lambda t} \int_0^t e^{-\lambda \tau} g(\tau) d\tau\]
provided we can do the actual integration! Now we have to solve \((D-\mu I)y = w\) and the integrating factor method leads to

\[ y(t) = e^{\mu t} \int_0^t e^{-\sigma}w(\sigma) d\sigma. \]

If you feel in need of practice in integration, just go back and re-calculate each of the above particular solutions by using this repeated integration method. Even though we can’t carry out the actual integrals, the argument guarantees that there is a particular solution as long as \(g(t)\) has no singularities. When we can’t carry out the integration and seek recourse to numerical approximations, this repeated integration formula is not a good way to find a particular solution. We shall give later another formula that involves only one integration, and that is much better numerically.

Notice also that we get the same conclusions for initial value problems for the equation \(Ly = g\). Check for yourself!

Now let’s go back to our original differential equation \(y'' + \alpha y' + \beta y = 0\) but allow \(\alpha\) and \(\beta\) to be functions of \(t\). In applications, this allows us to use more sophisticated models for our physical problems. But there is a price to pay — we no longer have a simple algorithm to solve the differential equation and we are forced to introduce more and more new special functions (such as Bessel functions) to describe the solutions. A detailed discussion is left to the next course in differential equations, but let’s see how far we can get by mimicking what we did for constant coefficients. The key idea was to factorize the differential operator and reduce the problem to solving two first order differential equations.

To appreciate the difficulty for non-constant coefficients, let’s expand the differential formula

\[ (D - tI)(D - t^2 I)y. \]

We get

\[ D(Dy - t^2 y) - t(D - t^2)y = D^2 y - t^2 Dy - 2ty - tDy + t^3 y. \]

Notice that there is an extra piece “\(-2ty\)” that is not present in the constant coefficient case. In fact, the general case is easier to see. We want a factorization

\[ (D^2 + \alpha D + \beta I)y = (D - \lambda I)(D - \mu I)y. \]
How do we find $\lambda$ and $\mu$? We expand the right side to get
\[(D - \lambda I)(D - \mu I)y = (D^2 - (\lambda + \mu) D + (\lambda \mu - \mu') I)y\]
and so we need
\[\lambda + \mu = -\alpha, \quad \lambda \mu - \mu' = \beta.\]
Substitute for $\lambda$ and we are reduced to solving
\[\mu' = -\mu(\mu + \alpha) + \beta.\]
Unfortunately, this is not a linear first order equation and we do not have a simple algorithm to solve it. By our theory for first order equations we certainly get existence and uniqueness for initial value problems on some interval for $t$. (More sophisticated methods show that we get solutions throughout any interval on which $\alpha$ and $\beta$ are nice functions.)

But suppose we have found functions $\lambda$ and $\mu$ as above. To solve
\[(D - \lambda I)(D - \mu I)y = 0\]
we just let $w = (D - \mu I)y$. First, we solve $w' - \lambda w = 0$ by the integrating factor method, and then we solve $y' - \mu y = w$ by the integrating factor method. It’s easy in theory but essentially impossible in practice. Of course there are other methods to solve the equation, especially when $\alpha$ and $\beta$ are polynomial functions — we just look for a power series solution, but the details are messy and appear in a later course. Again in theory, we can solve $(D - \lambda I)(D - \mu I)y = g$ by solving two first order differential equations by the integrating factor method.

Despite the impracticality of these methods it is worthwhile to pursue the theoretical ideas a little further. If we could carry out the integrations to solve $y'' + \alpha y' + \beta y = 0$ we would end up with a solution of the form $y = C_1 y_1 + C_2 y_2$ where $C_1, C_2$ are arbitrary constants. To solve the initial value problem with given values for $y(p), y'(p)$, we simply calculate $y' = C_1 y'_1 + C_2 y'_2$, put $t = p$ in these two equations and solve for $C_1, C_2$. To guarantee that we can always get unique solution for $C_1, C_2$, we need to know that
\[y_1(p)y'_2(p) - y'_1(p)y_2(p) \neq 0.\]
This $2 \times 2$ determinant is called the Wronskian of $y_1, y_2$, and it is denoted by $W(y_1, y_2)$. So, if the Wronskian does not vanish at $t = p$ then we can solve any initial value problem starting from $t = p$. 

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But our Wronskian has the remarkable property that it has value zero at \( t = p \) if and only if it has value zero for all \( t \)! To see this, note that \( y_1, y_2 \) satisfy the equation \( y'' + \alpha y' + \beta y = 0 \) and then calculate to get \( W' = -\alpha W \). This gives \( W = Ae^{\gamma(t)} \) where \( \gamma' = -\alpha \). Since exponentials never vanish, our claim follows. (Caution: although \( e^{\log t} = t \) appears to vanish at \( t = 0 \), the function \( \log t \) is not nice at \( t = 0 \) and so cannot solve our differential equation at \( t = 0 \).) So the non-vanishing of the Wronskian for just one \( p \) implies that we can solve the initial value problem starting at any \( t \) value \( p \).

There is some other useful language to describe what is going on here. We say that two functions \( y_1, y_2 \) are **linearly dependent** if one of the functions is a constant multiple of the other. Otherwise, we say that \( y_1, y_2 \) are **linearly independent**; evidently this amounts to saying that we can have \( c_1 y_1 + c_2 y_2 = 0 \) if and only if both real coefficients \( c_1, c_2 \) are zero. If \( y_1, y_2 \) are linearly dependent, then we must have \( W'(y_1, y_2) = 0 \). For if \( c_1 y_1 + c_2 y_2 = 0 \) with \( c_1, c_2 \) not both zero, then we also get \( c_1 y'_1 + c_2 y'_2 = 0 \). By linear equation theory, the \( 2 \times 2 \) determinant of coefficients must vanish, in other words, the Wronskian is always zero. The converse is false for arbitrary functions \( y_1, y_2 \) (can you find such an example?); but the converse is true when \( y_1, y_2 \) are solutions of a differential equation \( y'' + \alpha y' + \beta y = 0 \).

To see this, let us suppose that \( y_1, y_2 \) are solutions of such a differential equation and that the Wronskian vanishes at some \( t = p \). By linear equation theory, we can find \( c_1, c_2 \) not both zero so that

\[
   c_1 y_1(p) + c_2 y_2(p) = 0, \quad c_1 y'_1(p) + c_2 y'_2(p) = 0.
\]

Let \( y = c_1 y_1 + c_2 y_2 \). Then \( y \) solves the same differential equation and has the initial values \( y(p) = y'(p) = 0 \). By the uniqueness of the solution to an initial value problem, \( y \) has to be the zero function; in other words, the functions \( y_1, y_2 \) are linearly dependent.

As a curious application of this, note that the functions \( t, t^2 \) are obviously linearly independent as functions on the real line. Their Wronskian vanishes at one point \( t = 0 \), but nowhere else, and hence they cannot be solutions of any nice differential equation of the form \( y'' + \alpha y' + \beta y = 0 \). On the other hand, they can be solutions of such an equation on the positive real line — check that they solve the differential equation

\[
t^2 y'' - 2ty' + 2y = 0.
\]

Notice that, for this differential equation, the functions \( \alpha = -2/t, \beta = 2/t^2 \) have singularities at \( t = 0 \).
Finally we shall give the promised second formula for a particular solution to the differential equation
\[ y'' + \alpha y' + \beta y = g(t). \]
We look for a solution of the form
\[ y = y_1 u_1 + y_2 u_2 \]
where \( y_1, y_2 \) are solutions of \( y'' + \alpha y' + \beta y = 0 \) and \( u_1, u_2 \) are functions yet to be determined. Why on earth should we try this (the method of variation of parameters)? Because it works! We calculate
\[ y' = (y_1' u_1 + y_2' u_2) + (y_1 u_1' + y_2 u_2'). \]
Let’s restrict our search for \( u_1, u_2 \) to those functions such that
\[ y_1 u_1' + y_2 u_2' = 0. \]
So the formula for \( y'' \) just becomes
\[ y'' = (y_1'' u_1 + y_2'' u_2) + (y_1' u_1' + y_2' u_2'). \]
It follows easily that
\[ y'' + \alpha y' + \beta y = y_1' u_1' + y_2' u_2', \]
and so \( y \) is a particular solution provided the functions \( u_1, u_2 \) also satisfy the equation
\[ y_1' u_1' + y_2' u_2' = g. \]
We now solve these two equations for \( u_1, u_2 \) to find that
\[ u_1' = -y_2 g / W(y_1, y_2), \quad u_2' = y_1 g / W(y_1, y_2). \]
We now integrate (in theory!) to find \( u_1, u_2 \) and hence our particular solution \( y = y_1 u_1 + y_2 u_2 \). As long as \( y_1, y_2 \) are linearly independent solutions, the Wronskian \( W(y_1, y_2) \) will never vanish. You are invited to write \( y \) as the integral of the quotient of two \( 2 \times 2 \) determinants. The formula looks pretty, but it is tiresome to calculate, even in the case when \( \alpha, \beta \) are constants. At least, it is not a repeated integral and so it is easy to handle numerically when we have formulas for \( y_1 \) and \( y_2 \).

We mention just two applications for these differential equations (but only in the case of constant coefficients). A mass \( m \) on a spring is acted on by
gravity and Hooke’s law. The displacement $y$ from the equilibrium position satisfies an equation of the form $y'' + \omega^2 y = 0$ and has independent solutions $\cos \omega t, \sin \omega t$. Every solution is of the form $A \cos (\omega t - \theta)$ and the motion is called a pure harmonic oscillation. When there is a resistance to motion (for example, when the spring is in a bath of oil) that is directly proportional to the speed, the equation becomes $y'' + \kappa y' + \omega^2 y = 0$ with $\kappa > 0$. The roots of the characteristic equation are now either both negative or have negative real parts, and so the motion dies out to zero. When we have complex roots, we call the motion a damped oscillation.

We get exactly the same mathematical story with simple series electrical circuits. With an inductance $L$ and a capacitor $C$ in series, by Kirchoff’s Laws the charge $Q$ satisfies the equation $LQ'' + Q/C = 0$ and so we get a pure harmonic oscillation. When we add in a resistor $R$ in the same series, the equation becomes $LQ'' + RQ' + Q/C = 0$ and again the charge dies out to zero (with a damped oscillation in the case of complex roots). When we put several circuits together in parallel, the story becomes much more complicated — it will be considered later under the theme of systems of linear differential equations.

Suppose now that we introduce an external forcing term into a pure harmonic oscillation. If the forcing term has the same frequency as the harmonic oscillator, for example, $y'' + y = \cos t$, then a particular solution includes a $t$ term, $y = (1/2)t \sin t$ in this case. So the oscillations increase wildly until the apparatus fails. This is called resonance and it can lead to the destruction of suspension bridges. We get a similar situation for an $LC$ circuit so that we can greatly increase the output of the circuit. In the presence of damping, we have negative exponentials and so the input of a $t$ term fails to stop the eventual decay to zero of the solutions.

Everything that we have said for second order linear differential equations can be extended to third order, fourth order, . . . For the homogeneous equation we get an obvious generalization of the characteristic equation and the general solution is a sum of exponential terms. If the root $\lambda$ is repeated $k - fold$ then the contribution to the general solution is given by

$$e^{\lambda t}(C_1 + C_2 t + C_3 t^3 + \cdots + C_k t^{k-1})$$

as you may easily verify by solving the equation $(D - \lambda I)^k y = 0$ by our usual technique! The same algorithm works for particular solutions, and all the Wronskian ideas also generalize.
PROBLEMS 2

1. Find the relationship between the complex coefficients \(C_1, C_2\) and the real coefficients \(A_1, A_2\) when
\[
C_1e^{(a+ib)t} + C_2e^{(a-ib)t} = e^{at}(A_1 \cos bt + A_2 \sin bt)
\]

2. Solve
(i) \(y'' - y = 0; \quad y(0) = 0, y(1) = 0\)
(ii) \(y'' + 2y' + 2y = 0; \quad y(0) = 1, y(\pi) = 0\)
(iii) \(y'' + 8y' - 9y = 0; \quad y(1) = 1, y'(1) = 0\)
(iv) \(y'' + 8y' - 9y = 0; \quad y(0) = -9, y'(0) = 0\)

3. Find the general solution of
(i) \(y'' + 2y' + 2y = t^2\)
(ii) \(y'' - y' - 2y = te^{-t}\)
(iii) \(y'' + 4y' + 4y = t^2e^{-2t}\)
(iv) \(y'' + y = t + \sin t\)

4. Solve
\[
y'' + y = \cos \omega t; \quad y(0) = 0, y'(0) = 0
\]
for the cases (i) \(\omega = 0.95\), (ii) \(\omega = 1\). For general \(\omega > 0\) denote the solution by \(y_\omega\). Is it true that \(y_\omega(t) \to y_1(t)\) as \(\omega \to 1\)?

5. Given \(\omega > 0\), solve
\[
y'' + 2y' + 2y = e^{-t}\cos \omega t; \quad y(0) = y'(0) = 0
\]
for the cases (i) \(\omega \neq 1\), (ii) \(\omega = 1\). What happens to the solution as \(\omega \to 1\)?

6. In terms of the constants \(L, R, C\), determine the rate at which \(Q(t) \to 0\) as \(t \to +\infty\) when \(LQ'' + RQ' + Q/C = 0\).

7. Find the general solution of
\[
(D^4 - 2D^2 + I)y = e^t + \sin t
\]

8. Find a particular solution of
\[
(D^3 - D)y = t \sinh t + \cosh t
\]
3 THE LAPLACE TRANSFORM

The Laplace Transform is a wonderful tool to convert calculus problems to algebraic problems; the only hitch is that some of the algebraic reformulations are non-trivial. In particular, when we solve ordinary differential equations using the Laplace Transform, we often wonder if we would not be better (and quicker) to stick with the earlier “sausage-machine” algorithm. Happily, the Laplace Transform has lots of other uses besides ordinary differential equations — and we have to admit that it has marvellous mathematical properties. The fundamental idea is to transform a function \( y(t) \) into a new function \( Y(s) \) by the equation:

\[
Y(s) = \int_0^\infty e^{-st} y(t) dt.
\]

Of course this integral will not converge if \( y(t) \) grows too quickly at infinity, e.g. \( y(t) = \exp(t^2) \). Happily, in many situations our function \( y(t) \) is of exponential growth in the sense that \( |y(t)| \leq Me^{at} \). It is then easy to show that the integral converges provided \( s > a \). In many situations, \( y', y'', \ldots \) are also of exponential growth. Integration by parts now gives the glorious formula:

\[
\int_0^\infty e^{-st} y'(t) dt = sY(s) - y(0).
\]

Thus differentiation with respect to \( t \) corresponds (roughly) to multiplication by \( s \) — algebra instead of calculus! Repeat the process and we get

\[
\int_0^\infty e^{-st} y''(t) dt = s^2Y(s) - sy(0) - y'(0).
\]

It is convenient to denote the Laplace Transform by \( \mathcal{L} \), thus \( Y(s) = \mathcal{L}(y) \). It is obvious that \( \mathcal{L} \) is linear in that

\[
\mathcal{L}(ay + bz) = a\mathcal{L}(y) + b\mathcal{L}(z)
\]

where \( a, b \) are constants and \( y, z \) functions of \( t \). What happens if we take the Laplace Transform of the differential equation \( y'' + y = 0? \) We get

\[
s^2Y(s) - sy(0) - y'(0) + Y(s) = 0
\]
and hence

\[ Y(s) = \frac{sy(0) + y'(0)}{s^2 + 1}. \]

To solve our differential equation, all we have to do is find out which function \( y(t) \) has the above \( Y(s) \) as its Laplace Transform. As long as \( y(t) \) is a “nice” function, there is only one formula \( y(t) \) that transforms to give \( Y(s) \) — but this fact is NOT obvious and is quite hard to prove. Given that fact, our strategy is now clear. We want a dictionary to be able to translate back and forth between \( y(t) \) and \( Y(s) \). Unfortunately, this gets to be complicated (compare the difficulty of integration as against differentiation).

As a first step, let’s calculate the Laplace Transform of our favorite functions (especially for differential equations). We get

\[
\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}
\]

and so, in particular, \( \mathcal{L}(1) = \frac{1}{s} \). Here, we are thinking of \( a \) as real, but it works the same if \( a \) is pure imaginary, say \( a = ib \). This gives

\[
\mathcal{L}(e^{ibt}) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2}.
\]

Take real and imaginary parts to get

\[
\mathcal{L}(\cos(bt)) = \frac{s}{s^2+b^2}, \quad \mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2}.
\]

We could continue with more complicated functions \( y(t) \), but it pays instead to get some rules for the Laplace Transform.

In the original equation for \( Y(s) \), what happens if we differentiate with respect to \( s \)? For our nice functions of exponential growth, it is permissible to “differentiate under the integral sign” and we get the marvellous formula:

\[
Y'(s) = \int_0^\infty (-t)e^{-st}y(t)dt.
\]

In other words, this says that \( \mathcal{L}((-t)y) = Y'(s) \). In particular, we get \( \mathcal{L}(-t) = (d/ds)(1/s) = -1/s^2 \). Then we get \( \mathcal{L}((-t)^2) = (d/ds)(-1/s^2) = 2/s^3 \). Continue in this fashion and we get the general formula:

\[
\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.
\]
A similar easy computation gives
\[ \mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}} \]
and also
\[ \mathcal{L}(t \cos(bt)) = \frac{-1}{s^2 + b^2} + \frac{2s^2}{(s^2 + b^2)^2}, \quad \mathcal{L}(t \sin(bt)) = \frac{2bs}{(s^2 + b^2)^2}. \]
With higher powers of \( t \), the trig transforms become more and more complicated!

We have already found the Laplace Transforms for almost all of the functions that appear as solutions of most second order linear differential equations with constant coefficients. There is one more simple rule that gives an easy computation for the missing formulas. Notice that
\[ \mathcal{L}(e^{at} y(t)) = \int_0^\infty e^{-st} e^{at} y(t) dt = \int_0^\infty e^{-(s-a)t} y(t) dt = Y(s-a). \]
In particular, we get
\[ \mathcal{L}(e^{at} \cos(bt)) = \frac{s-a}{(s-a)^2 + b^2}, \quad \mathcal{L}(e^{at} \sin(bt)) = \frac{b}{(s-a)^2 + b^2}. \]
These are enough formulas for the present. Let’s return to differential equations again. For the equation \( y'' + y = 0 \) we got
\[ Y(s) = y(0) \frac{s}{s^2 + 1} + y'(0) \frac{1}{s^2 + 1}. \]
Since the inverse Laplace Transform \( \mathcal{L}^{-1} \) is also linear, we see immediately that
\[ y(t) = y(0) \cos(t) + y'(0) \sin(t). \]
This agrees with our solution in the last chapter, except that we now have explicit values for the arbitrary constants \( A, B \) in terms of the initial values \( y(0), y'(0) \). [What happens if the initial values are given at \( t = p \)? Then we have to write \( z(t) = y(t+p) \), find the corresponding differential equation for \( z \), solve it by the Laplace transform, and then use \( y(t) = z(t-p) \).]

We might as well consider other generic examples. Suppose \( y'' - y = 0 \). Take the Laplace Transform to get
\[ s^2 Y(s) - sy(0) - y'(0) - Y(s) = 0, \quad Y(s) = \frac{sy(0) + y'(0)}{s^2 - 1}. \]
To make life easier, let’s say that \( y(0) = 1, y'(0) = 2 \). This gives

\[
Y(s) = \frac{s + 2}{s^2 - 1} = \frac{s + 2}{(s - 1)(s + 1)}.
\]

By the “cover-up” rule for partial fractions we get

\[
Y(s) = \frac{3/2}{s - 1} + \frac{-1/2}{s + 1}.
\]

From our earlier list of Laplace Transforms, we recognize these transforms and we get

\[
y(t) = \left(\frac{3}{2}\right)e^t - \left(\frac{1}{2}\right)e^{-t}.
\]

Suppose now that \( y'' - 2y' + y = 0 \) and \( y(0) = 1, y'(0) = 2 \). Take the Laplace Transform and do some algebra to get

\[
Y(s) = \frac{s}{(s - 1)^2} = \frac{s - 1 + 1}{(s - 1)^2} = \frac{1}{s - 1} + \frac{1}{(s - 1)^2}.
\]

From the list of transforms above, we see that

\[
y(t) = e^t + te^t.
\]

Suppose now that \( y'' - 2y' + 2y = 0 \) and \( y(0) = 1, y'(0) = 2 \). Take the Laplace Transform and do some algebra to get

\[
Y(s) = \frac{s}{s^2 - 2s + 2} = \frac{s - 1}{(s - 1)^2 + 1} + \frac{1}{(s - 1)^2 + 1}.
\]

From the list of transforms above, we see that

\[
y(t) = e^t \cos(t) + e^t \sin(t).
\]

For the general case

\[
y'' + \alpha y' + \beta y = 0
\]

we get

\[
(s^2 + \alpha s + \beta)Y(s) = y(0)s + y'(0) + \alpha y(0).
\]

Notice that the characteristic quadratic immediately appears as the coefficient of \( Y(s) \). We just complete the square for the characteristic quadratic
and then do all the algebra. This seems almost as easy as the “sausage machine” method, but let’s see what happens when we introduce a forcing term on the right hand side.

Consider the equation \( y'' + y = e^{-t} \cos(t) \) and let’s take the nicest possible initial values, namely, \( y(0) = y'(0) = 0 \). Take the Laplace Transform and we get

\[
(s^2 + 1)Y(s) = \frac{s + 1}{(s + 1)^2 + 1}.
\]

This time, the partial fraction exercise is non-trivial. We have

\[
\frac{s + 1}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 2}.
\]

Cross-multiply to give

\[
s + 1 = (As + B)(s^2 + 2s + 2) + (Cs + D)(s^2 + 1).
\]

Put \( s = i \) to give

\[
i + 1 = (Ai + B)(1 + 2i), \quad B + iA = (1 + i)(1 - 2i)/5 = (3 - i)/5.
\]

Hence \( A = -1/5, B = 3/5 \). We can now put \( s + 1 = i \) and get another complex number equation for \( C, D \). Alternatively, we can equate coefficients of \( s^0 \) and \( s^0 \) to get \( A + C = 0 \) and \( 2B + D = 1 \). This gives \( C = 1/5 \) and \( D = -1/5 \). This leads to

\[
Y(s) = \frac{-1}{5} \frac{s}{s^2 + 1} + \frac{3}{5} \frac{1}{s^2 + 1} + \frac{1}{5} \frac{s + 1}{(s + 1)^2 + 1} + \frac{-2}{5} \frac{1}{(s + 1)^2 + 1}
\]

and hence

\[
y(t) = -(1/5) \cos(t) + (3/5) \sin(t) + (1/5)e^{-t} \cos(t) - (2/5)e^{-t} \sin(t).
\]

But this is hardly easier than the “sausage machine” method! If you need more convincing, just try replacing the above forcing term with \( te^{-t} \cos(t) \).

There are some simple strategies which can simplify the calculation of partial fractions. To handle \( 1/[(s^2 + 1)(s^2 + 2)] \) we may let \( \sigma = s^2 \). The “cover-up” rule gives

\[
\frac{1}{(\sigma + 1)(\sigma + 2)} = \frac{1}{\sigma + 1} - \frac{1}{\sigma + 2}.
\]
Now replace $\sigma$ by $s^2$. To handle $s/[(s^2 + 1)(s^2 + 2)]$ we do the above example and just multiply through by $s$.

After wrestling through a few messy partial fraction computations, one naturally asks if there is a way to avoid them. Yes there is, but it just puts the hard work somewhere else, in terms of integration instead of algebra. Consider the general case

$$y'' + \alpha y' + \beta y = g(t); \quad y(0) = y'(0) = 0.$$ 

Apply the Laplace Transform to get

$$(s^2 + \alpha s + \beta)Y(s) = G(s)$$

and hence $Y(s) = F(s)G(s)$ where $F(s) = 1/(s^2 + \alpha s + \beta)$. We can easily work out the inverse Laplace Transform of $F(s)$, say $f(t)$, and we know the inverse Laplace Transform of $G(s)$, namely, $g(t)$. Is there a formula for the inverse Laplace Transform of $F(s)G(s)$ in terms of $f(t)$ and $g(t)$? Yes! It is called the convolution of $f$ and $g$ and is defined by

$$f \ast g(t) = \int_0^t f(t - \tau) g(\tau) d\tau.$$ 

To prove this, we use interchange of order of integration and change of variables:

$$\int_0^\infty e^{-st} f \ast g(t) dt = \int_0^\infty \int_0^t e^{-st} f(t - \tau) g(\tau) d\tau dt$$

$$= \int_0^\infty g(\tau) d\tau \int_\tau^\infty e^{-su} f(t) dt$$

$$= \int_0^\infty g(\tau) d\tau \int_0^{s\tau} e^{-u} f(u) du$$

$$= \int_0^\infty e^{-su} f(u) du \int_0^\infty e^{-st} g(\tau) d\tau$$

$$= F(s)G(s).$$

Now let’s apply this to the equation $y'' + y = e^{-t} \cos(t); y(0) = y'(0) = 0$. We saw that the Laplace Transform was given by

$$Y(s) = \frac{1}{s^2 + 1} \frac{s + 1}{(s + 1)^2 + 1}.$$
and so the convolution theorem gives us

\[ y(t) = \int_0^t \sin(t - \tau)e^{-\tau} \cos(\tau) d\tau \]

\[ = (1/2) \int_0^t e^{-\tau}[\sin(t) + \sin(t - 2\tau)] d\tau. \]

where we have used a trig identity. This integral is not trivial, but it will lead eventually to the same formula as above. You may consider that this is not much of an improvement over the partial fractions. In fact we have seen this formula in the last chapter; the integral formula for a particular integral (with the Wronskian in the denominator) turns out to be exactly the above formula. Well, more power to the “sausage machine” method!

Despite the above example, the convolution theorem does make it easier to calculate a few inverse Laplace Transforms. For example,

\[ \mathcal{L}^{-1} \left( \frac{1}{s(s^2 + 1)} \right) = \int_0^t \sin(t) d\tau = 1 - \cos(t) \]

\[ \mathcal{L}^{-1} \left( \frac{1}{(s^2 + 1)^2} \right) = \int_0^t \sin(t - \tau) \sin(\tau) d\tau \]

\[ = (1/2) \int_0^t [\cos(t - 2\tau) - \cos(t)] d\tau \]

\[ = (1/2) \sin(t) - (1/2)t \cos(t). \]

\[ \mathcal{L}^{-1} \left( \frac{1}{(s^2 + 1)(s^2 + 4)} \right) = \int_0^t \sin(t - \tau) \sin(2\tau) d\tau \]

\[ = (1/2) \int_0^t [\cos(t - 3\tau) - \cos(t + \tau)] d\tau \]

\[ = (4/3) \sin(t) - (2/3) \sin(2t). \]

Surprisingly, there are situations where the Laplace Transform is quicker than the “sausage machine” method. This happens when the forcing term is switched on for a short period of time and then switched off. Consider the equation \( y'' + y = g(t); \) \( y(0) = y'(0) = 0, \) where \( g(t) = 1 \) for \( 0 \leq t \leq \pi \) and \( g(t) = 0 \) for \( t > \pi. \) Solve this by the “sausage machine” method on the
interval $[0, \pi]$ to get $y = 1 + A \cos(t) + B \sin(t)$. Apply the initial values to get $y = 1 - \cos(t)$. This gives $y(\pi) = 2$ and $y'(\pi) = 0$. We now solve $y'' + y = 0$ for $t > \pi$ with these values at $\pi$ to get $y = -2 \cos(t)$ for $t > \pi$.

Notice that we have forced the solution to be continuous and differentiable at $t = \pi$. But, of course, $y''$ has a jump discontinuity at $t = \pi$. [Recall that such examples occur in dynamics when a particle moving at uniform speed around a circle suddenly goes off along a tangent line. In geometrical language, the curvature changes suddenly from a positive constant to zero.]

Now we consider the Laplace Transform viewpoint on this problem. Notice that we can write the above $g(t)$ as $g(t) = 1 - u_\pi(t)$ where $u_\pi$ is a Heaviside function. [The Heaviside function is introduced here because it gives nice Laplace transform formulas.] For any real $c$, we define the Heaviside function $u_c$ by $u_c(t) = 0$ for $0 \leq t \leq c$ and $u_c(t) = 1$ for $t > c$. Notice that

$$\mathcal{L}(u_c) = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}.$$  

More generally, we have

$$\mathcal{L}(u_c(t)y(t-c)) = \int_c^\infty e^{-st} y(t-c) dt = e^{-cs}Y(s).$$

Given $y'' + y = 1 - u_\pi; y(0) = y'(0) = 0$ we quickly get

$$Y(s) = \frac{1}{s(s^2 + 1)} - \frac{e^{-\pi s}}{s(s^2 + 1)}$$

and it follows immediately that

$$y(t) = 1 - \cos(t) - u_\pi(t)[1 - \cos(t - \pi)].$$

We easily check that this is the same answer as above. More generally, if the forcing term $g(t)$ is applied for $0 \leq t \leq c$ and then switched off, we simply replace the right hand side of the equation by $g(t) - u_c(t)g(t)$ and proceed as above.

It is very helpful to make up one’s own table of the key rules for the Laplace Transform together with the fundamental formulas for the most important functions in applications. We have now covered all the functions that we have encountered in our earlier solution of second order linear equations.
with constant coefficients. There are a few odds and ends that we can also pick up at this time. First, we calculate a few more Laplace Transforms.

Earlier, we considered $L(t^n)$ only when $n$ is a non-negative integer. More generally we can find the Laplace Transform of $t^p$ for any real $p > -1$ by a simple change of variable.

$$L(t^p) = \int_0^\infty e^{-st}t^p dt = \int_0^\infty e^{-u}u^p \frac{du}{sp},$$

This gives $L(t^p) = \Gamma(p+1)/s^{p+1}$ where the Gamma function is defined by

$$\Gamma(p+1) = \int_0^\infty e^{-u}u^p du.$$ When $p$ is a positive integer, we have $\Gamma(p+1) = p!$. The Gamma function was introduced as a smooth interpolant for the factorial function on the positive integers. Notice that we need the restriction $p > -1$ or else our integrals become divergent at the origin! When $p = -1/2$ we get the very interesting formula

$$L(1/\sqrt{t}) = \Gamma(1/2)(1/\sqrt{s}).$$

It is a nice review exercise from Calculus III to show that $\Gamma(1/2) = \sqrt{\pi}$. The above formula is of special interest because the Laplace Transform is the same as the original formula apart from the constant $\sqrt{\pi}$. Since the Gamma function obeys the rule $\Gamma(p+1) = p\Gamma(p)$, it is easy to calculate the Laplace Transform of $t^{n+1/2}$.

The function $\sin(t)/t$ (and its translates by multiples of $\pi$) is important in the theory of signal processing. Let’s call it $\eta(t)$, with Laplace Transform $H(s)$. Notice that $(-t)\eta(t) = -\sin(t)$ and so

$$H'(s) = L((-t)\eta(t)) = \frac{1}{s^2 + 1}.$$ It follows that $H(s) = C - \arctan(s)$. How are we to find $C$? It is easy to see that, for any function $y(t)$ of exponential growth, we have $Y(s)$ converging to 0 as $s$ converges to $+\infty$. It follows immediately that $C = \pi/2$ and so

$$H(s) = \pi/2 - \arctan(s) = \arctan(1/s).$$

We have applied the Laplace Transform only to second order linear differential equations with constant coefficients. We can also use the Laplace
Transform when the coefficients are polynomials. We illustrate the idea only very briefly. Consider the Airy equation $y'' - ty = 0; y(0) = y'(0) = 0$. Apply the Laplace Transform and we get $s^2Y(s) + Y'(s) = 0$. This is a first order equation that we can solve for $Y(s)$ — but the formula is not nice! The study of such differential equations with polynomial coefficients is the subject of another course!

In the much more difficult subject of \textit{partial differential equations} (such as Laplace’s equation discussed in Calculus III), the Laplace Transform can sometimes be used to reduce the partial differential equation (say, in two variables) to an ordinary differential equation (in one variable).
PROBLEMS 3

1. Solve

\[ y'' + y = 0; \quad y(p) = a, \; y'(p) = b \]

by setting \( z(t) = y(t + p) \), as suggested in the text.

2. Find

(i) \( L(te^{-t} \sin t) \)
(ii) \( L(te^t \cos 2t) \)
(iii) \( L(t^2 \sin 3t) \)
(iv) \( L((t - t^2) \cos 3t) \)

3. Solve

\[ y'' + y = te^{-t} \cos t; \quad y(0) = y'(0) = 0 \]

4. Solve

\[ y'' - y' + 2y = e^t \sin t; \quad y(0) = 0, y'(0) = 0 \]

5. Find

(i) \( L^{-1}\left(\frac{1}{s^2 + 5}\right) \)
(ii) \( L^{-1}\left(\frac{1}{s^2 + 9}\right) \)
(iii) \( L^{-1}\left(\frac{1}{(s^2 + 1)(s^2 + s + 1)}\right) \)
(iv) \( L^{-1}\left(\frac{1}{(s^2 + 4)^3}\right) \)

6. Solve

\[ y'' + y = g(t); \quad y(0) = y'(0) = 0 \]

where \( g(t) = \sin t \) for \( 0 \leq t \leq \pi/2 \) and \( g(t) = 0 \) for \( t > \pi/2 \)

7. Solve

\[ y'' + 2y' + y = g(t); \quad y(0) = y'(0) = 0 \]

where \( g(t) = t^2 \) for \( 0 \leq t \leq \pi \) and \( g(t) = 0 \) for \( t > \pi \)

8. Solve

\[ y'' - y = g(t); \quad y(0) = y'(0) = 0 \]

where \( g(t) = t \) for \( 0 \leq t \leq 1 \), \( g(t) = e^t \) for \( 1 < t \leq 2 \) and \( g(t) = 0 \) for \( t > 2 \)
4 LINEAR SYSTEMS

In more complicated modeling problems we have more than one unknown function, and the rate of change of each unknown function depends on all of the other unknown functions. A typical example comes from a predator-prey situation in ecology. In the simplest case there is one species of predator and one species of prey, and their populations at time $t$ give the two unknown functions. In more sophisticated modern examples we may have hundreds or even thousands of unknown functions.

For most of this chapter, our linear systems of differential equations will be homogeneous with constant coefficients. In the later part we shall include forcing terms and non-constant coefficients.

We note first that we can build first order systems of differential equations from one higher order differential equation. Consider the second order equation

$$x'' + 2x' - 5x = 0.$$ 

Introduce two unknown functions by

$$x_1 = x, \quad x_2 = x'.$$

Then we have $x'_2 = x'' = 5x - 2x' = 5x_1 - 2x_2$. Thus our one second order differential equation is equivalent to the system

$$x'_1 = x_2, \quad x'_2 = 5x_1 - 2x_2.$$ 

Similarly, one third order differential equation is equivalent to a system of three first order equations; and so on. Note also that a system of second order differential equations can be reduced to a larger system of first order differential equations.

There is a simple method for $2 \times 2$ systems, but unfortunately it becomes impractical for bigger systems. For a $2 \times 2$ system it is easy to eliminate one of the unknown functions. Consider a simple example:

$$x'_1 = x_1 + x_2, \quad x'_2 = 2x_1 - 3x_2.$$ 

We can rewrite these equations as

$$(D - 1)x_1 = x_2, \quad (D + 3)x_2 = 2x_1.$$
This gives
\[(D + 3)(D - 1)x_1 = (D + 3)x_2 = 2x_1\]
and hence we get \((D^2 + 2D - 5)x_1 = 0\). We already know how to solve this equation! We can get an explicit solution provided we know the initial values \(x_1(0)\) and \(x'_1(0)\). The initial values for our given system are provided by the two numbers \(x_1(0)\) and \(x_2(0)\), and then \(x'_1(0)\) is given by the equation \(x'_1(0) = x_1(0) + x_2(0)\). So we can find \(x_1\). We can find \(x_2\) by a similar method (and it satisfies the same second order differential equation!), but clearly it is quicker to find \(x_2\) from the equation \(x_2 = x'_1 - x_1\).

Clearly the above method can be applied to any system:

\[x'_1 = px_1 + qx_2, \quad x'_2 = rx_1 + sx_2.\]

For our second approach, we want to rewrite these two equations as one equation! We can achieve this by introducing vectors and matrices. It is common practice to write vectors horizontally as \(\mathbf{x} = \langle x_1, x_2 \rangle\) (where \(x_1, x_2\) may be scalars or functions). It is here more convenient to write them vertically as

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

The derivative of a vector function is defined as usual, namely,

\[
\mathbf{x}' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}.
\]

The two equations can now be written as one equation

\[
\mathbf{x}' = A\mathbf{x}
\]

where \(A\) is the matrix given by

\[
A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}
\]

and where the multiplication \(A\mathbf{x}\) is specified by the two given equations. We can also say it as follows. The first entry of \(A\mathbf{x}\) is the dot product of the first row of \(A\) with the vector \(\mathbf{x}\), and the second entry of \(A\mathbf{x}\) is the dot product of the second row of \(A\) with the vector \(\mathbf{x}\). [This definition extends to the case when \(\mathbf{x}\) has \(n\) components and \(A\) is an \(n \times n\) matrix.] The advantage of
this notation is that a system of 100 equations is still written by the single equation \( x' =Ax \). We may now regard the initial values for the system to be specified by the vector \( x(0) \).

For a \( 1 \times 1 \) system \( x' = ax \) we know that the unique solution is given by \( x = x(0)e^{at} \), or, as we now prefer to write it, \( x = e^{ta}x(0) \). Could it be that the unique solution to the general system \( x' = Ax \) is given by \( x = e^{tA}x(0) \)? This is indeed correct, but we have to explain what we mean by the exponential of a matrix, why this gives the unique solution, and how we can calculate it.

For a real number \( t \) we know that \( e^t = 1 + t + t^2/2! + \cdots \). We have also seen that the infinite series gives a good meaning for \( e^z \) when \( z \) is a complex number. For a square matrix \( C \) we should like to define the exponential by

\[
e^C = I + C + C^2/2! + C^3/3! + \cdots
\]

Here \( I \) is the identity matrix that has 1 at each place down the dexter diagonal and 0 everywhere else. What do we mean by \( C^2, C^3, \) and so on? We consider the \( 2 \times 2 \) case. Let \( A, B \) be \( 2 \times 2 \) matrices. We can write \( B \) as two vertical vectors, \( B = [b_1, b_2] \). Notice that \( b_1 = Bi, b_2 = Bj \) where \( i, j \) are the usual basis vectors of 2-space. Motivated by a repeated linear change of variables (as in Calculus III) we now define \( AB \) to be the \( 2 \times 2 \) matrix given by

\[
AB = [Ab_1 \ Ab_2].
\]

[Recall that we have already defined how to multiply a matrix into a vector.] Notice that \( AI = A = IA \), so that the matrix \( I \) behaves like the number 1. We define \( C^2 = CC, C^3 = CC^2, \) and so on. An infinite series of \( 2 \times 2 \) matrices produces four infinite series — one for each position of the \( 2 \times 2 \) matrix. We can then say that the infinite series of matrices converges if the same is true for each of the four associated infinite series. We can take it on trust that the series for \( e^C \) converges for \textit{any} matrix \( C \).

What do we mean by \( tA \) (or, more generally, \( f(t)A \))? We just mean the matrix in which every entry in \( A \) is multiplied by \( t \) (or, more generally, by \( f(t) \)). It is easy to see that we now have the equation

\[
e^{tA} = I + tA + t^2A^2/2! + t^3A^3/3! + \cdots
\]
and so we have now given a meaning to the formula $e^{tA}x(0)$. Notice that $e^{tA} = I$ when $t = 0$, so that our formula has the required value when $t = 0$. Now we have to check that our formula satisfies the equation $x' = Ax$.

We know how to differentiate a vector function, but how do we differentiate a matrix function? And then how do we differentiate the product of a matrix function and a vector function? We differentiate a matrix function just by differentiating each function inside the matrix. So the derivative of $tA$ is $A$, and the derivative of $t^2A$ is $2tA$, and so on. We take on trust that the infinite series for $e^{tA}$ can be differentiated term by term and so we get

$$\frac{d}{dt}e^{tA} = 0 + A + tA^2 + t^2A^3/2! + \cdots.$$ 

We can factor out $A$ on either side (this is important!) to get the unsurprising formula

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A.$$ 

You may check for yourself that the product rule for differentiating a matrix times a vector is just the usual one. Thus we get

$$\frac{d}{dt}e^{tA}x(0) = Ae^{tA}x(0) + e^{tA}\frac{d}{dt}x(0) = Ae^{tA}x(0)$$

so that $x = e^{tA}x(0)$ is a solution of the equation $x' = Ax$ with initial values $x(0)$.

Next we show that the solution is unique. Suppose that $x$ solves the equation $x' = Ax$ with initial values $x(0)$. Let $z = e^{-tA}x$ and then the product rule gives

$$z' = -Ae^{-tA}x + e^{-tA}x' = -Ae^{-tA}x + e^{-tA}Ax = 0$$

and so $z$ is constant with value $z(0) = x(0)$. So we have proved that $e^{-tA}x = x(0)$. Multiply on the left by $e^{tA}$ and we get $x = e^{tA}x(0)$, as required. [Why is it true that $e^{tA}e^{-tA} = I$? Just differentiate each side with respect to $t$, using the usual product rule to differentiate the product of two matrices.]

**HOW DO WE CALCULATE $e^{tA}$?**

We have done the easy part — we have found the formula for the solution. Now we come to the hard part — calculate $e^{tA}$. For a general matrix $A$ it is
a formidable exercise to calculate every $A^n$ and then sum the power series. But there are a few special kinds of matrices for which it is easy to do so. Suppose that we have a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$ 

It is easy to calculate that

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

and then more generally that

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.$$ 

It follows from the infinite series that

$$e^{t\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$$ 

Is there something we can do to a general matrix $A$ to turn it into a diagonal matrix $\Lambda$? There are several ways to motivate the method; here is one way. Notice that the diagonal matrix $\Lambda$ satisfies the equations

$$\Lambda i = \lambda_1 i, \quad \Lambda j = \lambda_2 j$$

where $i, j$ are the usual basis vectors of 2-space. For a general matrix $A$ we might look for vectors $c_1, c_2$ such that

$$Ac_1 = \lambda_1 c_1, \quad Ac_2 = \lambda_2 c_2.$$ 

We say that $\lambda_1, \lambda_2$ are eigenvalues of $A$ and that $c_1, c_2$ are corresponding eigenvectors. [In older books, these are called characteristic roots and characteristic vectors, or latent roots and latent vectors.] We do not regard the zero vector as an eigenvector. Note that if $c_1$ is an eigenvector, then so also is $kc_1$ for any non-zero scalar $k$. We often find it convenient to normalize an eigenvector by taking one of the entries to be 1.

Let $P$ be the matrix formed by the two eigenvectors:

$$P = [c_1\ c_2].$$
Now we get
\[ AP = [A\mathbf{c}_1 \ A\mathbf{c}_2] = [\lambda_1\mathbf{c}_1 \ \lambda_2\mathbf{c}_2] = P\Lambda. \]

It often happens that the matrix \( P \) is invertible, that is, there is a matrix \( P^{-1} \) such that \( PP^{-1} = I = P^{-1}P \). Multiply the above equation on the left by \( P^{-1} \) and we get the wonderful equation
\[ P^{-1}AP = \Lambda. \]

So we have converted the matrix \( A \) into a diagonal matrix \( \Lambda \)! Now notice that
\[ \Lambda^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P. \]

More generally, we get
\[ \Lambda^n = P^{-1}APP^{-1}AP \ldots = P^{-1}A^nP. \]

Now it is easy to see that
\[ P^{-1}e^{tA}P = e^{t\Lambda}. \]

But we have already seen that it is easy to calculate \( e^{t\Lambda} \). Cross-multiply and we get finally that
\[ e^{tA} = Pe^{t\Lambda}P^{-1}. \]

This calculates \( e^{tA} \) for us. The entries in \( P \) and \( P^{-1} \) are scalars, and so we see that the entries in \( e^{tA} \) are just scalar combinations of the functions \( e^{\lambda_1t} \) and \( e^{\lambda_2t} \).

Now we are reduced to the following problem. Given a matrix \( A \), how do we calculate the eigenvalues and corresponding eigenvectors? We have to find the values of \( \lambda \) for which we can find a non-zero vector \( \mathbf{c} \) with \( A\mathbf{c} = \lambda\mathbf{c} \). We can rewrite this equation as \( B\mathbf{c} = \mathbf{0} \) where \( B = A - \lambda I \). For the \( 2 \times 2 \) case it is easy to check that we can get a non-zero \( \mathbf{c} \) with \( B\mathbf{c} = \mathbf{0} \) if and only if \( \det B = 0 \). We recall from Calculus III that
\[ \det \begin{bmatrix} p & q \\ r & s \end{bmatrix} = ps - qr. \]

[By linear algebra, this is true also in the \( n \times n \) case, though the definition of determinant is complicated.] In the \( 2 \times 2 \) case, the equation \( \det(A - \lambda I) = 0 \) is just a quadratic equation and so we always get two roots \( \lambda = \lambda_1, \lambda_2 \). For these two values of \( \lambda \) we solve the linear system \((A - \lambda I)\mathbf{c} = \mathbf{0}\) to find corresponding normalized eigenvectors \( \mathbf{c}_1, \mathbf{c}_2 \). We can always do this when
\(\lambda_1 \neq \lambda_2\) (we discuss the other case later) and then the matrix \(P = [c_1 \ c_2]\) is always invertible so that we can complete our calculation of \(e^{tA}\). For a \(2 \times 2\) matrix, there is a very simple formula for the inverse matrix as follows:

\[
P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad P^{-1} = \frac{1}{\det P} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}.
\]

We now illustrate with some numerical examples. Let

\[
A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.
\]

We get the eigenvalues by solving \((2 - \lambda)^2 - 1 = 0\) so that \(\lambda = 3, 1\). For \(\lambda = 3\) we have to solve

\[
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

This gives \(c_1 = c_2\) and we normalize by taking \(c_1 = 1\). For \(\lambda = 1\) we have to solve

\[
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

This gives \(c_1 + c_2 = 0\) and we normalize by taking \(c_2 = 1\). This gives

\[
P = [c_1 \ c_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

and hence

\[
e^{tA} = Pe^{tA}P^{-1} = \frac{1}{2} \begin{bmatrix} e^{3t} + e^t & e^{3t} - e^t \\ e^{3t} - e^t & e^{3t} + e^t \end{bmatrix}.
\]

We chose the numbers in the above example to make life easy for ourselves. If we replace one of the 2’s in \(A\) by a 1, we get the eigenvalues to be \((3 \pm \sqrt{5})/2\) and all the rest of the formulas become messy. In fact, any matrix that is symmetric about the dexter diagonal (such a matrix is said to be symmetric) always has real eigenvalues and we can always get enough eigenvectors to complete the process — but the computations may be very messy. Now we take an example with complex eigenvalues.

Let

\[
A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.
\]

We get the eigenvalues by solving \((1 - \lambda)^2 + 1 = 0\), so that \(\lambda = 1 \pm i\). Notice that the eigenvalues are of the form \(\lambda\) and \(\lambda^*\), the complex conjugate of \(\lambda\).
Thus, if $Ac = \lambda c$, then by taking complex conjugates we get $Ac^* = \lambda^* c^*$. In other words, as soon as we find an eigenvector for $\lambda$, we immediately have one for $\lambda^*$. For $\lambda = 1 + i$ we have to solve
\[
\begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]
This gives $c_2 = ic_1$, and we may take $c_1 = 1$. Now we take
\[
P = \begin{bmatrix}
c_1 & c_2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
i & -i
\end{bmatrix}, \quad P^{-1} = \frac{-1}{2i} \begin{bmatrix}
-i & -1 \\
-i & 1
\end{bmatrix}
\]
and this gives
\[
e^{tA} = P \begin{bmatrix}
e^{(1+i)t} & 0 \\
0 & e^{(1-i)t}
\end{bmatrix} P^{-1}.
\]
After some computation, this reduces to
\[
e^{tA} = \begin{bmatrix}
e^t \cos t & e^t \sin t \\
-e^t \sin t & e^t \cos t
\end{bmatrix}.
\]
Again we have made the numbers easier than usual.

Finally we consider an example with a repeated eigenvalue, in which the above algorithm fails. Let
\[
A = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.
\]
We get the eigenvalues by solving $(1 - \lambda)^2 = 0$ so that $\lambda = 1$ (twice). Now we have to solve $(A - I)c = 0$ and this gives only $c_2 = 0$. We normalize by taking $c_1 = 1$. It is now impossible to find an invertible $P$ with $AP = PA$ and so the above algorithm fails. But it turns out to be very easy to calculate $e^{tA}$ in this case. We can write $A = I + N$ where $IN = NI$ and $N^2 = 0$. It follows that $(I + N)^2 = I^2 + 2NI + N^2 = I + 2N$. Similarly we get $(I + N)^3 = I + 3N$ and so on. This gives
\[
e^{tA} = I + t(I + N) + (t^2/2!)(I + 2N) + (t^3/3!)(I + 3N) + \cdots = e^t(I + tN).
\]
This method works quite generally. If the eigenvalues are $\lambda$ (twice), then it turns out that either we get $P^{-1}AP = \lambda I$ and so $A = \lambda I$, or else $P^{-1}AP = \lambda I + N$ with $N^2 = 0$. But then we have
\[
A = P(\lambda I + N)P^{-1} = \lambda I + PNP^{-1}.
\]
Notice that
\[ PNP^{-1}PNP^{-1} = PNP^{-1} = P = 0. \]
This gives \( A = \lambda I + M \) with \( M^2 = 0 \) and hence \( e^{tA} = e^{\lambda t}(I + tM) \). In practice, we get the repeated eigenvalue \( \lambda \) and put \( M = A - \lambda I \). We can check that \( M^2 = 0 \) and then we just write down the above formula for \( e^{tA} \). Notice how this links with the second order case of repeated roots when we got \( e^{\lambda t} \) and \( te^{\lambda t} \) as linearly independent solutions.

We have now covered all the cases that arise for \( 2 \times 2 \) matrices \( A \). For \( 3 \times 3 \) or larger matrices, the same principles apply but the computational effort increases dramatically. For a polynomial of degree \( n \), we know that there are \( n \) roots (some or all complex). But for \( n \) large it is a very difficult (and numerically unreliable) task to determine them. Moreover, for each eigenvalue it is a serious task to calculate a corresponding normalized eigenvector. There is still active research going on for good algorithms to calculate \( e^{tA} \) when \( A \) is a matrix of large size.

But we ought to mention a few special cases in which it is very easy to calculate \( e^{tA} \). Every \( n \times n \) matrix satisfies a polynomial equation of degree at most \( n \). As a special case we can check easily that every \( 2 \times 2 \) matrix satisfies the equation
\[ A^2 - tr(A)A + \det(A)I = 0 \]
where \( tr(A) \) is the sum of the elements on the dexter diagonal. Suppose that \( A^2 = A \). Then \( A^k = A \) for every positive integer \( k \) and so we get
\[ e^{tA} = I + tA + t^2 A/2! + t^3 A/3! + \cdots = I + (e^t - 1)A. \]
Suppose now that \( A^2 = I \). Then \( A^3 = A, A^4 = I \), and so on. Hence
\[ e^{tA} = I + tA + t^2 I/2! + t^3 A/3! + \cdots = (\cosh t)I + (\sinh t)A. \]
Suppose now that \( A^2 = -I \). It is easy to check that
\[ e^{tA} = (\cos t)I + (\sin t)A. \]
We remark in passing that each of these special quadratic equations has an infinite number of different solutions \( A \)!
IS THERE ANOTHER WAY TO FIND $e^{tA}$?

It is reasonable to ask whether there are any other shortcuts to find $e^{tA}$. Note that the first column of $e^{tA}$ is given by $e^{tA}i$, and so that column is just the solution of $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = i$. Similarly we get the second column of $e^{tA}$ by solving $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = j$. How then can we solve $\mathbf{x}' = A\mathbf{x}$ with initial values $\mathbf{x}(0)$ without actually calculating $e^{tA}$. We know how to this this bare hands for the $2 \times 2$ case, but that method usually becomes too complicated for bigger matrices. There is a halfway house available in which we just calculate the eigenvalues and eigenvectors but do not have to go on to calculate $P^{-1}$. Suppose that we have eigenvalues $\lambda_1, \lambda_2$ with $\lambda_1 \neq \lambda_2$. We easily verify that $e^{\lambda_1 t} c_1$ is a solution of $\mathbf{x}' = A\mathbf{x}$ provided $c_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1$. As usual we can normalize the eigenvector $c_1$. Now we easily check that

$$\mathbf{x} = C_1 e^{\lambda_1 t} c_1 + C_2 e^{\lambda_2 t} c_2$$

is a solution of $\mathbf{x}' = A\mathbf{x}$ for any constants $C_1, C_2$. As usual, we can determine these constants by setting $t = 0$ and then solving the vector equation

$$C_1 c_1 + C_2 c_2 = \mathbf{x}(0).$$

Notice that we have thus provided another method to solve $\mathbf{x}' = A\mathbf{x}$ with initial vector $\mathbf{x}(0)$. By doing the solution for two initial vectors we can thus find $e^{tA}$ (in the $2 \times 2$ case).

The situation is rather more complicated when $\lambda_1 = \lambda_2$ and we can find only one normalized eigenvector $c$. It is tempting to look for a second solution of the form $te^{\lambda_1 t} u$. In fact, this does not work and we have to go on to look for a solution of the form $te^{\lambda_1 t} u + e^{\lambda_1 t} v$. In other words, the method we gave above for finding $e^{tA}$ in this case is much quicker!

HOW DO WE HANDLE FORCING TERMS?

We turn now to the situation when a forcing term is present. Thus we want to solve the equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}.$$

Suppose that $y$ is a particular solution. Let $z$ be any solution of the equation $\mathbf{z}' = A\mathbf{z}$. We easily check that that $\mathbf{x} = y + z$ gives a solution of the equation $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$. On the other hand, if $\mathbf{x}$ is any solution of the equation...
\( \mathbf{x}' = A\mathbf{x} + \mathbf{g} \), then we easily check that \( \mathbf{z} = \mathbf{x} - \mathbf{y} \) is a solution of the equation 
\( \mathbf{z}' = A\mathbf{z} \). Thus we have proved that the \textit{general} solution of the equation 
\( \mathbf{x}' = A\mathbf{x} + \mathbf{g} \) is given by the sum of a particular solution \( \mathbf{y} \) and the general 
solution of the equation \( \mathbf{x}' = A\mathbf{x} \). Notice the clear parallel with the result for 
the general solution of a second order linear differential equation with forcing 
term.

How then can we find a particular solution? If we have gone to the 
trouble of finding \( \mathbf{P} \) with \( \mathbf{P}^{-1}A\mathbf{P} = \Lambda \) with \( \Lambda \) a diagonal matrix, then it is 
plain sailing. We just write \( \mathbf{x} = \mathbf{P}\mathbf{y} \) and then we have \( \mathbf{x}' = \mathbf{P}\mathbf{y}' \). Substitute 
to get
\[
\mathbf{P}\mathbf{y}' = A\mathbf{P}\mathbf{y} + \mathbf{g}, \quad \mathbf{y}' = \Lambda\mathbf{y} + \mathbf{P}^{-1}\mathbf{g}.
\]
This last system of equations is de-coupled — only one unknown occurs in 
each equation and hence we can solve for each unknown by the method of 
chapter 1. Once we have found \( \mathbf{y} \), we get \( \mathbf{x} \) from the equation \( \mathbf{x} = \mathbf{P}\mathbf{y} \). For 
repeated eigenvalues in a large matrix we treat each Jordan block by the 
method below.

Next, we shall give a formula — but the nature of the formula will prompt 
us to look for a shorter method for natural examples of the forcing term \( \mathbf{g} \).

Recall from the first chapter that we were able to solve the single equation 
\( \mathbf{x}' = a\mathbf{x} + g(t) \) 
by using the integrating fact \( e^{-at} \). We can mimic that here, except that we 
need the exponential of a matrix. Rewrite our equation as \( \mathbf{x}' - A\mathbf{x} = \mathbf{g} \) and 
multiply both sides by \( e^{-tA} \) to get the equivalent equation
\[
\frac{d}{dt}[e^{-tA}\mathbf{x}] = e^{-tA}\mathbf{g}.
\]
Hence, one solution is given by
\[
e^{-tA}\mathbf{x} = \int_0^t e^{-\tau A}\mathbf{g}(\tau)d\tau.
\]
Multiply both sides by \( e^{tA} \) and we get
\[
\mathbf{x} = e^{tA}\int_0^t e^{-\tau A}\mathbf{g}(\tau)d\tau.
\]
It is easy to see that we can move the term $e^{tA}$ inside the integral sign and so we end up with the convolution formula for a particular solution:

$$x = \int_0^t e^{(t-\tau)A} g(\tau) d\tau.$$

[In fact we easily see that the general solution is given by

$$x = e^{tA} x(0) + \int_0^t e^{(t-\tau)A} g(\tau) d\tau.$$]

but if we just want one particular solution we may as well take $x(0) = 0$.]

This is a nice simple formula, but we recall how much effort it may need to calculate the exponential of a matrix. Suppose that the forcing term is a combination of exponentials (including cosine and sine terms) and polynomials. Can we mimic the “sausage machine” process for second order linear differential equations? Let’s try.

Suppose first that $g$ is of the form $e^{ct} h$. Clearly we ought to try the substitution $x = e^{ct} w$. This gives

$$x' = ce^{ct} w + e^{ct} w'.$$

The equation $x' = Ax + g$ now changes to the equation

$$e^{ct}[cw + w'] = e^{ct}[Aw + h].$$

We can cancel off the exponential terms and we are reduced to the equation

$$w' = (A - cI)w + h.$$

This is an equation of similar type but with the exponential term missing. Suppose that the remaining forcing term $h$ is a polynomial. For simplicity, let us take

$$h = tu + v$$

where $u, v$ are constant vectors.

Thus we want to find a particular solution of an equation of the type

$$x' - Bx = tu + v$$

where $B = A - cI$. The “sausage machine” algorithm tells us to differentiate until a solution becomes obvious. Differentiate once to get

$$x'' - Bx' = u.$$
We should like to take \( x' \) to be a constant vector with \(-Bx' = u\) and then we shall have \( x'' = 0\); then we back substitute into the original equation to find \( x \). But there is a hitch. We cannot always solve the equation \( Bb = u \). To guarantee a solution we need to know not that \( B \) is not the zero matrix (as in the scalar case), but that \( B \) is invertible. When \( B \) is invertible we can certainly complete the back substitution to find a particular solution. The same method applies (with more work) for polynomials of higher order giving the forcing term \( h \).

How do we cope with the case when \( B \) is not invertible? This is troublesome in general, but it is easy in the \( 2 \times 2 \) case. If \( B \) is not invertible, then \( \det(B) = 0 \) and so \( B \) satisfies the equation \( B^2 = kB \) where \( k \) is the trace of \( B \). By elaborating an earlier argument we find that

\[
e^{tB} = I + e^{kt} - \frac{1}{k} B
\]

and now it is easy to use our general convolution formula to give a particular solution. Of course, it can happen that we can start with an invertible matrix \( A \) but after we remove the exponential term we find that \( B = A - cI \) is not invertible – in which case, we use this last method. This situation will arise precisely when \( c \) is one of the eigenvalues of \( A \).

Now we illustrate all of this with some (easy) numerical examples. Let’s find a particular solution of \( x' = Ax + g \) when

\[
A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad g = e^{2t} \begin{bmatrix} t \\ 2 \end{bmatrix}.
\]

First we set \( x = e^{2t}w \) and we are reduced to solving

\[
w' = (A - 2I)w + h
\]

where \( h = <t, 2> \) (written horizontally). We note that

\[
A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}
\]

and this is invertible with inverse matrix given by

\[
(A - 2I)^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.
\]
Differentiation gives
\[ w'' = (A - 2I)w' + h' \]
and so we get a solution with
\[ w' = -\frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
Now we back substitute to get
\[ (A - 2I)w = h - w' \]
and hence
\[ w = -\frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} t - 1/2 \\ 3/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} t - 2 \\ t + 1 \end{bmatrix}. \]
Of course, \( x = e^{2t}w. \)
Now let’s find a particular solution of \( x' = Ax + g \) when
\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} t \\ 2 \end{bmatrix}. \]
Since \( A^2 = 0 \) we have \( e^{\xi A} = I + \xi A \) and so
\[ e^{(t-\tau)A}g(\tau) = \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau \\ 2 \end{bmatrix} = \begin{bmatrix} 2t - \tau \\ 2 \end{bmatrix}. \]
So, a particular solution is given by
\[ x = \int_0^t e^{(t-\tau)A}g(\tau)d\tau = \begin{bmatrix} 3t^2/2 \\ 2t \end{bmatrix}. \]
We leave it as an exercise to find the general solution in each of the above examples. Of course, the work becomes much heavier if \( A \) is \( 3 \times 3 \) or larger.
We can also solve systems by the (vector) Laplace transform. Write
\[ X(s) = \int_0^\infty e^{-st}x(t)dt \]
and we easily see that the equation
\[ x' = Ax + g \]
transforms to
\[ s \mathbf{X}(s) - \mathbf{x}(0) = A \mathbf{X}(s) + \mathbf{G}(s) \]
or,
\[ \mathbf{X}(s) = (sI - A)^{-1}[\mathbf{x}(0) + \mathbf{G}(s)]. \]
The entries in the matrix \((sI - A)^{-1}\) are rational functions of \(s\). Of course we get \(x(t)\) by finding the inverse Laplace transform of each entry in \(\mathbf{X}(s)\). But it’s sometimes easier said than done — we leave you to try some \(2 \times 2\) examples for yourself (preferably ones for which we already know the answer).

We should note here that we have produced another formula for \(e^{tA}\), namely \(\mathcal{L}^{-1}((sI - A)^{-1})\) (by which we mean, of course, the matrix in which each entry is the inverse Laplace Transform of the corresponding entry in \((sI - A)^{-1}\). To see this, just put \(g = 0\) in the above discussion, and take \(\mathbf{x}(0)\) to be \(\mathbf{i}\) and then \(\mathbf{j}\) to get the first and second columns of of \(e^{tA}\) to be the first and second columns of \(\mathcal{L}^{-1}((sI - A)^{-1})\).

Finally we turn to the situation where the constant matrix \(A\) is replaced by a matrix of functions \(P(t)\). The theoretical discussion of the solutions is almost the same as before, but the actual calculation of solutions is usually hopeless. We begin as usual with the homogeneous case: \(\mathbf{x}' = P(t) \mathbf{x}\). In the scalar case and in the constant matrix case, we solved this by the integrating factor method. We can do the same again, at least in some cases. We rewrite the equation as \(\mathbf{x}' - P(t) \mathbf{x} = 0\) and then we multiply by \(e^{Q(t)}\) to make the left side an exact derivative. We want
\[ e^{Q(t)} \mathbf{x}' - e^{Q(t)} P(t) \mathbf{x} = \frac{d}{dt} e^{Q(t)} \mathbf{x}. \]
It seems that we need to have
\[ e^{Q(t)} Q'(t) = -e^{Q(t)} P(t) \]
or, in other words \(Q'(t) = -P(t)\). This last equation is easily achieved — we just take an antiderivative for each function inside the matrix \(P(t)\). But there is a catch! How do we know that
\[ \frac{d}{dt} e^{Q(t)} = e^{Q(t)} Q'(t)? \]

The exponential is defined by the usual power series formula and we want to differentiate term by term. Let’s try the term \(Q^2\) (for simplicity we shall
drop the \( t \)'s. The product rule gives

\[
\frac{d}{dt}Q^2 = Q'Q + QQ'.
\]

But matrix multiplication is not commutative and so, in general, we cannot write this as \( 2QQ' \). Similarly we have

\[
\frac{d}{dt}Q^3 = Q'QQ + QQ'Q + QQQ'.
\]

Thus, in general, the formula for the derivative of \( e^Q \) is just a terrible mess! We can get the above formula for the derivative provided we have \( QQ' = Q'Q \).

For example, if \( Q = tA \), then \( Q' = A \) and we do have \( QQ' = Q'Q \). More generally, if \( Q = f(t)A \), then again we get \( QQ' = Q'Q \). There are lots of cases when we get \( QQ' = Q'Q \), but if we write down a matrix function at random we shall not have \( QQ' = Q'Q \). When we do have \( QQ' = Q'Q \), then our system has solution

\[
e^{Q(t)}x = c
\]

where \( c \) is a constant vector. We can always choose our antiderivatives so that \( Q(0) = 0 \). Put \( t = 0 \) and we find \( c = x(0) \). Thus our system has unique solution

\[
x = e^{-Q(t)}x(0).
\]

Now we come to the really serious problem. How do we calculate \( e^{-Q(t)} \)? In general, we have no good method to do this. Does this mean that we have come to the end of the road for the effectiveness of the integrating factor method? Yes.

There are other methods that will show the existence and uniqueness of solution for \( x' = P(t)x \) for given initial values \( x(0) \). But they still do not give us effective methods to calculate the actual solutions. We can study properties of the solutions. For example, we can generalize our earlier results about linear independence of solutions. In particular, given an \( n \times n \) system with solutions \( x_1, \ldots, x_n \) we define the associated \textit{Wronskian} by

\[
W = \det[x_1, \ldots, x_n].
\]

Linear algebra shows that the \( n \) solutions are linearly independent if and only if \( W \) is not identically zero; in which case, it follows (arguing as in Chapter 2) that the general solution is just a linear combination of these \( n \) independent
solutions. It is a nice exercise (in the $2 \times 2$ case) to prove, for independent solutions, that $W' = tr(P)W$ and hence $W$ is either identically zero or else it is never zero. But we shall not pursue the details since we do not have effective methods to calculate such solutions.

Afterthought: in this chapter we have already used a few special cases of the general result below. We can’t resist giving two proofs of this important result.

Let $A, B$ be square matrices with $AB = BA$. Then $e^A e^B = e^{A+B}$.

Proof by “algebra”. To calculate $e^{A+B}$ from the usual infinite series we need to know $(A + B)^n$. Notice that


In general no further simplification is possible. But when $AB = BA$ we get the usual formula

$$(A + B)^2 = A^2 + 2AB + B^2$$

and then we get

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

and so on for all the higher powers. To get $e^A e^B$ we multiply together the two power series and gather together the terms of the same degree. Thus we get

$$e^A e^B = (I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots)(I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \cdots).$$

The terms of degree 2 give

$$\frac{1}{2!} B^2 + AB + \frac{1}{2!} A^2 = \frac{1}{2!} (A^2 + 2AB + B^2)$$

the terms of degree 3 give

$$\frac{1}{3!} B^3 + \frac{1}{2!} AB^2 + \frac{1}{2!} A^2 B + \frac{1}{3!} A^3 = \frac{1}{3!} (A^3 + 3A^2B + 3AB^2 + B^3)$$

and so on. This gives $e^A e^B = e^{A+B}$.  

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Proof by “calculus”. We shall prove that $e^{tA}e^{tB} = e^{(A+B)t}$ for all $t$. (In the $2 \times 2$ case) it is enough to prove that these matrices gives the same answer when multiplied into $i$ and $j$. The argument is the same in each case. Note that $e^{(A+B)t}i$ is the unique solution of $x' = (A + B)x$ with $x(0) = i$. It is enough to show that $x = e^{tA}e^{tB}i$ gives another such solution. By the product rule we get

$$x' = Ae^{tA}e^{tB}i + e^{tA}Be^{tB}i.$$  

It is easy to see that $e^{tA}B = Be^{tA}$ (because $AB = BA$) and hence we get

$$x' = (A + B)e^{tA}e^{tB}i = (A + B)x.$$  

When $t = 0$, we get $e^{tA}e^{tB}i = i$. This completes the proof.
PROBLEMS 4

1. Solve
   (i) \( x'_1 = x_2, \quad x'_2 = 3x_1 + 2x_2; \quad x_1(0) = 1, x_2(0) = 0 \)
   (ii) \( x'_1 = x_1 + 2x_2, \quad x'_2 = 2x_1 + 3x_2; \quad x_1(0) = 0, x_2(0) = 1 \)
   (iii) \( x'_1 = x_1 + x_2, \quad x'_2 = -2x_1 + x_2; \quad x_1(0) = 1, x_2(0) = 1 \)

2. Calculate \( e^{tA} \) when
   (i) \( A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \)
   (ii) \( A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \)
   (iii) \( A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \)
   (iv) \( A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \)

3. Calculate \( e^{tA} \) when
   (i) \( A^2 = kA \), \quad (ii) \( A^2 = k^2I \), \quad (iii) \( A^2 = -k^2I \).

4. Find a particular solution for the system
   (i) \( x' = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x + e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).
   (ii) \( x' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + t^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
   (iii) \( x' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} x + te^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).
   (iv) \( x' = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} e^t \\ -t \end{bmatrix} \).
   (v) \( x' = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x + e^{3t} \begin{bmatrix} 1 \\ t \end{bmatrix} \).

5. For each \( A \) in example 2 above, calculate \((sI-A)^{-1}\). Hence, recalculate \( e^{tA} \) in each case, using the inverse Laplace Transform.
5 DIFFERENCE EQUATIONS

Difference equations arise, in particular, by replacing derivatives by differences. At a very crude level we can think of approximating \( f'(x) \) by the Newton quotient \( \Delta f(x) = f(x + 1) - f(x) \). Thus, the differential equation

\[
Df(x) = af(x)
\]

would be replaced by the difference equation

\[
\Delta f(x) = af(x).
\]

This difference equation can be rewritten as \( f(x + 1) - f(x) = af(x) \) or even, \( f(x + 1) = (1 + a)f(x) \). It is natural to restrict the variable \( x \) to take the values \( 0, 1, 2, \ldots \). It is then common notation to replace \( f(n) \) by \( x_n \). Our difference equation now gets rewritten as a recurrence

\[
x_{n+1} = (1 + a)x_n.
\]

It is very easy to see that this difference equation has general solution \( x_n = (1 + a)^n x_0 \).

More realistically, we might replace \( f'(x) \) by \( \frac{f(x + h) - f(x)}{h} \) where \( h \) is small. This replaces a differential equation by a difference equation which can be solved by recurrence. This is the starting point for the numerical solution of differential equations; but this simplistic idea has to be replaced by more sophisticated algorithms to get reliable numerical solutions. We shall not pursue this idea here — a whole course is needed for it.

Returning to our original theme, we may replace \( D^2 f(x) \) by

\[
\Delta^2 f(x) = \Delta [f(x + 1) - f(x)] = f(x + 2) - 2f(x + 1) + f(x).
\]

This is called the (forward) second difference of \( f(x) \). The second order differential equation

\[
(D^2 + \alpha D + \beta I)f(x) = 0
\]

would now be replaced by the difference equation

\[
(\Delta^2 + \alpha \Delta + \beta I)f(x) = 0.
\]

At the sequence level this converts to the recurrence

\[
x_{n+2} + (\alpha - 2)x_{n+1} + (1 - \alpha + \beta)x_n = 0.
\]
Given any starting values $x_0, x_1$, we clearly get a unique value for each subsequent $x_n$. And it is easy to program a computer to calculate these values. It is easy to pass backwards and forwards between a difference operator equation and a recurrence equation. In practice we work with the recurrence equations although we can use the difference operator equation to motivate methods of solution, by analogy with differential equations.

Difference equations in the form of recurrences come up in many mathematical settings; for example in counting problems. Given $n$ numbers, in how many different ways can we insert brackets to indicate the order in which the multiplications are to be carried out? Or, more easily, in how many different ways, say $F_n$, can we tile a $2 \times n$ board with $n$ dominoes? Focus on how we cover the upper last square and we see that $F_n$ satisfies the recurrence

$$F_{n+1} = F_n + F_{n-1}$$

and we easily check that $F_1 = 1$ and $F_2 = 2$. This determines each integer $F_n$ — but it does not tell us how rapidly these numbers grow. To get information on the order of magnitude of $F_n$ we want to have a closed formula for $F_n$ (the $n^{th}$ Fibonacci number).

By analogy with differential equations we should expect a solution of the form $e^{kn}$. Replace $e^k$ by $\lambda$ and we are looking for a solution of the form $F_n = \lambda^n$. Substitute this formula into the equation and we see that we need to have

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1}.$$ 

Since we have no interest in the case $\lambda = 0$, we can divide out to get the quadratic equation

$$\lambda^2 = \lambda + 1.$$ 

This has solutions $\lambda = \lambda_1 = [1 + \sqrt{5}]/2$, $\lambda = \lambda_2 = [1 - \sqrt{5}]/2$. It follows, as in differential equations (linearity!) that $C_1\lambda_1^n + C_2\lambda_2^n$ gives a solution of the difference equation for any values of $C_1$, $C_2$. We can put $n = 1, 2$ to get two equations to solve for $C_1$ and $C_2$. In this case we find (after some work) that

$$F_n = \frac{1}{\sqrt{5}}[\lambda_1^{n+1} - \lambda_2^{n+1}].$$

Despite appearance, each $F_n$ is a whole number (it counts something!). The second term in the formula converges to zero and so the first term gives the order of magnitude of $F_n$. 

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Now we are ready to consider the general second order linear difference equation with constant coefficients
\[ x_{n+1} + \alpha x_n + \beta x_{n-1} = 0 \]
and we suppose we are given the initial values of \( x_0 \) and \( x_1 \). [It helps the algebra to be able to start at \( n = 0 \)!] Argue as above to see that \( x_n = \lambda^n \) (we are not interested in the case \( \lambda = 0 \)) gives a solution if and only if
\[ \lambda^2 + \alpha \lambda + \beta = 0. \]
Of course we call this the *characteristic equation* for the difference equation.
Solve to get \( \lambda = \lambda_1 \) and \( \lambda = \lambda_2 \). By linearity we check that
\[ x_n = C_1 \lambda_1^n + C_2 \lambda_2^n \]
gives a solution for any choice of constants \( C_1 \) and \( C_2 \).

**Case 1:** \( \lambda_1 \neq \lambda_2 \). To see that the above gives the general solution, it is enough to show that we can always choose \( C_1 \) and \( C_2 \) to achieve the given initial values. We have to solve
\[
\begin{align*}
C_1 + C_2 &= x_0 \\
C_1 \lambda_1 + C_2 \lambda_2 &= x_1.
\end{align*}
\]
We can easily solve explicitly to get
\[
C_1 = \frac{\lambda_2 x_0 - x_1}{\lambda_2 - \lambda_1}, \quad C_2 = \frac{\lambda_1 x_0 - x_1}{\lambda_1 - \lambda_2}.
\]
When the roots are real, the order of magnitude of \( x_n \) is determined by the larger absolute value of the two roots. Suppose the roots are complex. Since we are assuming that \( \alpha \) and \( \beta \) are real, we then get \( \lambda_1 = a + ib \) and \( \lambda_2 = a - ib \). If we expand \((a + ib)^n\) by the binomial theorem we get a huge formula. Instead, we change to polar form to get \( a + ib = re^{i\theta} \) where \( r = |\lambda_1| \) and \( \theta \) is uniquely determined (modulo \( 2\pi \)) by the equations \( \cos(\theta) = a/r \) and \( \sin(\theta) = b/r \). Now we get
\[
\lambda_1^n = r^n [\cos(n\theta) + i \sin(n\theta)], \quad r^n [\cos(n\theta) - i \sin(n\theta)]
\]
and the usual argument shows that we can write
\[ x_n = r^n [R_1 \cos(n\theta) + R_2 \sin(n\theta)] \]
where \( R_1 \) and \( R_2 \) are arbitrary real constants. When \( \theta \) is a rational multiple of \( 2\pi \), say, \( \theta = 2\pi m/N \), the trigonometric part of the formula is periodic with period \( N \). For the other values of \( \theta \), the term \( \cos(n\theta) \) wanders apparently randomly up and down the interval \([-1, 1]\).

**Case 2:** \( \lambda_1 = \lambda_2 = \gamma \). Clearly the above formula gives essentially only one solution and we have to find a second solution. By analogy with differential equations, it is easy to guess that a second solution will be given by \( x_n = n\gamma^n \). We now check this. Note that the characteristic equation has to be \((\lambda - \gamma)^2 = 0\) and so the actual difference equation is

\[
x_{n+1} - 2\gamma x_n + \gamma^2 x_{n-1} = 0.
\]

Now we see that \( x_n = n\gamma^n \) is a solution provided

\[
(n + 1)\gamma^{n+1} - 2\gamma n\gamma^n + \gamma^2 (n - 1)\gamma^{n-1} = 0.
\]

This last equation is clearly true. Now we see that

\[
x_n = (C_1 + nC_2)\gamma^n
\]

is a solution for any constants \( C_1 \) and \( C_2 \). To see that we have our general solution we just have to check that we can always solve the equations

\[
C_1 = x_0
\]

\[
(C_1 + C_2)\gamma = x_1.
\]

This is obvious.

We now have a “sausage machine” algorithm to solve any difference equation of the form \( x_{n+1} + \alpha x_n + \beta x_{n-1} = 0 \).

We now consider the case when we have a forcing term on the right hand side

\[
x_{n+1} + \alpha x_n + \beta x_{n-1} = g(n).
\]

By the usual linearity argument we show that the general solution of this equation is given by the sum of one particular solution of this equation and the general solution of the corresponding homogeneous equation (with 0 on the right hand side). We shall restrict attention to the case in which \( g(n) = p(n)\gamma^n \) where \( p(n) \) is a polynomial function. As with differential equations, our first step is to reduce the problem to the case when only a polynomial is
present. This is very easy; the general method is adequately illustrated by a specific example:

\[ x_{n+1} - 2x_n + 2x_{n-1} = (n + 2)3^n. \]

Let \( x_n = 3^n w_n \) and substitute into the above equation to get

\[ 3^{n+1} w_{n+1} - 2 3^n w_n + 2 3^{n-1} w_{n-1} = (n + 2)3^n. \]

Divide through by \( 3^n \) and we get

\[ 3w_{n+1} - 2w_n + (2/3)w_{n-1} = n + 2 \]

and so we have reduced the problem to solving a polynomial case.

The polynomial case is a little trickier than in the differential equation setting where we differentiated until any answer was obvious then back-substituted and took antiderivatives as required. The process of taking antidifferences is not very nice, even for polynomials — find a sequence that differences to give \( n^3! \) Instead we shall just use judicious guess and check and we illustrate with a few examples.

Let's try \( x_n = an + b \). Substitute and rearrange terms to get

\[ (1 - 2 + 2)an + a + b - 2b - 2a + 2b = n + 2. \]

These sequences agree for all \( n \) and so we may equate coefficients to get \( a = 1 \)
and \( b = 3 \). Hence \( x_n = n + 3 \) is a particular solution.

Now let's change the equation to

\[ x_{n+1} + x_n - 2x_{n-1} = n + 2. \]

If we try \( x_n = an + b \) then we get the coefficient of \( n \) on the left hand side to be \( 1 + 1 - 2 = 0! \) So there is no solution of this form. Since the sum of the coefficients is 0 we notice that \( \lambda = 1 \) is a root of the characteristic equation; this contributes a term \( 1^n = 1 \) to the solution and so, by analogy with the differential equation case, we should look for a solution of the form \( x_n = an^2 + bn \). [We do not need to include a constant term since that is already part of any general solution!] Now when we substitute and gather terms together we get

\[ (1 + 1 - 2)an^2 + (2a + b + b + 2a - 2b)n + (a + b - 2a + 2b) = n + 2. \]
Equate coefficients again and we end up with $a = 1/4$ and $b = 3/4$; hence a particular solution is given by $x_n = (n^2 + 3n)/4$.

There is just one other case to consider. It may be that $\lambda = 1$ is a double root of the characteristic equation. But then the characteristic equation is given by $(\lambda - 1)^2 = 0$ and our difference equation is of the form

$$x_{n+1} - 2x_n + x_{n-1} = n + 2.$$  

Now we try for a solution of the form $x_n = an^3 + bn^2$ and after some work we get $x_n = (1/6)n^3 + n^2$.

What about difference equations with non-constant coefficients? The existence and uniqueness theorems are just as easy as above. But finding closed forms for the solutions or even the order of magnitude of the solutions for large $n$ can be a very difficult problem. But there is one very useful theorem due to Poincaré. Roughly, it says that if the variable coefficients are asymptotically constant, then the order of magnitude of the solutions is given by the order of magnitude of the solutions of the associated difference equation with the limiting values of the coefficients. For example, consider the difference equation:

$$x_{n+1} - (3 + \frac{2}{n})x_n + (1 - \frac{1}{n} + \frac{4}{n^2})x_{n-1} = 0.$$  

To get the order of magnitude of the solutions we just solve the associated difference equation:

$$y_{n+1} - 3y_n + y_{n-1} = 0.$$  

[The proof of Poincaré’s Theorem is hard work!]

As a final application of difference equations, we shall derive yet another formula for $e^{tA}$, at least in the $2 \times 2$ case. Recall that, for any $2 \times 2$ matrix $A$ we have $A^2 - \tau A + \delta I = 0$ where $\tau = tr(A)$ and $\delta = det A$. Thus we have $A^2 = \tau A - \delta I$ and hence

$$A^3 = \tau A^2 - \delta A = \tau(\tau A - \delta I) - \delta A = (\tau^2 - \delta)A - \tau \delta I.$$  

Continuing in this fashion we get $A^n = \alpha_n A + \beta_n I$, where the coefficients $\alpha_n, \beta_n$ have to be determined. Multiply the characteristic equation by $A^{n-1}$ and we get

$$A^{n+1} - \tau A^n + \delta A^{n-1} = 0$$  

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and so

$$(\alpha_{n+1}A + \beta_{n+1}I) - \tau(\alpha_nA + \beta_nI) + \delta(\alpha_{n-1}A + \beta_{n-1}I) = 0$$

or

$$(\alpha_{n+1} - \tau\alpha_n + \delta\alpha_{n-1})A + (\beta_{n+1} - \tau\beta_n + \delta\beta_{n-1})I = 0.$$ 

We may suppose without loss that $A$ and $I$ are linearly independent matrices, and so we conclude that

$$\alpha_{n+1} - \tau\alpha_n + \delta\alpha_{n-1} = 0, \quad \beta_{n+1} - \tau\beta_n + \delta\beta_{n-1} = 0.$$ 

The characteristic equation for these difference equations is just the characteristic equation of $A$.

Suppose first that the roots (i.e. eigenvalues!) $\lambda, \mu$ are distinct. Then we get

$$\alpha_n = a\lambda^n + b\mu^n; \quad \alpha_0 = 0, \alpha_1 = 1$$
$$\beta_n = c\lambda^n + d\mu^n; \quad \beta_0 = 1, \beta_1 = 0.$$ 

An easy computation leads to

$$\alpha_n = \frac{\lambda^n - \mu^n}{\lambda - \mu}, \quad \beta_n = \frac{-\mu\lambda^n + \lambda\mu^n}{\lambda - \mu}.$$ 

Since

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

we can now substitute for $A^n$. After some manipulation we arrive at the formula

$$e^{tA} = \beta_nI + \alpha_nA$$

or

$$e^{tA} = \frac{-\mu\lambda^n + \lambda\mu^n}{\lambda - \mu}I + \frac{\lambda^n - \mu^n}{\lambda - \mu}A$$

or

$$e^{tA} = \frac{e^{\lambda t}}{\mu - \lambda}(\mu I - A) + \frac{e^{\mu t}}{\lambda - \mu}(\lambda I - A).$$

When the roots are repeated $(\mu, \mu)$ we already know a simple formula, namely

$$e^{tA} = e^{\mu t}(I + t(A - \mu I)).$$
It is not hard to see that this formula also comes from letting $\lambda \to \mu$ in the above formula. We leave as an exercise the presentation of the real version of the above formula when the roots are $a + ib, a - ib$ — which formula may well be the simplest way to calculate $e^{tA}$ for the case of complex conjugate roots!
PROBLEMS 5

1. Solve the difference equation

(i) \( x_{n+1} + 5x_n + 6x_{n-1} = 0; \quad x_0 = 0, x_1 = 1 \)
(ii) \( x_{n+1} + 2x_n + 2x_{n-1} = 0; \quad x_0 = 1, x_1 = 0 \)
(iii) \( x_{n+1} + 2x_n + 3x_{n-1} = 0; \quad x_0 = 0, x_1 = 1 \)
(iv) \( x_{n+1} - 6x_n + 9x_{n-1} = 0; \quad x_0 = 1, x_1 = 1 \)

2. Find a particular solution of the difference equation

(i) \( x_{n+1} - x_{n-1} = 2^n \)
(ii) \( x_{n+1} - x_{n-1} = n + 1 \)
(iii) \( x_{n+1} + 2x_n + 2x_{n-1} = n^22^n \)
(iv) \( x_{n+1} - 2x_n - 3x_{n-1} = n3^n \)

3. Solve the difference equation

(i) \( x_{n+1} - x_{n-1} = n^2(-1)^n; \quad x_0 = x_1 = 0 \)
(ii) \( x_{n+1} + x_{n-1} = n2^n; \quad x_0, x_1 = 1 \)
(iii) \( x_{n+2} - 2x_n + x_{n-2} = n + 1; \quad x_0 = x_1 = x_2 = 0, x_3 = 1 \)

4. When the eigenvalues are \( \lambda = a + ib, \mu = a - ib \), express \( e^{tA} \) as a real linear combination of \( I \) and \( A \) in terms of \( e^{at}, \cos bt, \sin bt \).

5. Find \( e^{tA} \) in terms of \( I, A, A^2 \) when \( A \) is 3 \( \times \) 3 with distinct eigenvalues \( \lambda, \mu, \nu \).