0 INTRODUCTION

The Oklahoma-Arkansas section of the Mathematical Association of America has a special lecture each spring meeting in honor of Nathan Altshiller-Court who was an enthusiastic geometer at the University of Oklahoma in the earlier part of this century. Court was very concerned about the neglect of Euclidean Geometry in American Colleges. Virtually none was taught at college so that prospective teachers went back to high schools only with the knowledge that they had earlier received from high schools. For several years Court taught a new course on college geometry and then in 1925 published his successful text on the subject. His course was very well received and almost all teachers enrolled in it even though it was not compulsory. Court had two clear aims: first to ensure that teachers knew much more geometry than they would actually teach in the high school and secondly to give them a topic on which they could actually do research themselves. This course does not contain all the material in Court’s book (we have yet to catch up with the ’twenties) though I believe our problem list has more interesting ones than Court’s exercises. Our reduced content may not be inappropriate since the current high school geometry syllabus keeps getting farther behind what it was in the ’twenties.

Geometry is not a trivial subject - except by hindsight after hours of struggles. This is well illustrated in the old Scottish Higher Certificate Examination in Geometry. A student could earn 49% by being perfect in the “bookwork” questions (writing out proofs of standard theorems), but to pass the exam (> 50%) the student had to make some measurable progress in solving the unseen problems - it was recognized as unreasonable to expect the students to do the bulk of the unseen problems under exam conditions. [In this course the unseen problems will mainly be done under “take-home-test” conditions.] Our geometry problems are not one-liners - they require sustained effort and sustained reasoning. Students who believe that mathematics should just consist of memorizing and applying simple formulas would be well advised to avoid this course.

Although geometry is non-trivial, it is fascinating (once one gets absorbed in a geometry problem, it can be hard to quit). Many of the remarkable properties of triangles and circles never cease to amaze me. Students who like intellectual challenge and excitement should take this course. We shall not solve all the listed problems during the semester so there will be plenty of material for future self study. These notes form only a synopsis of the course.
Each student should write their own detailed course out of notes taken in the lectures - this is unquestionably the best way to get to grips with any subject, i.e. write your own text! Learners have to actualize the truth for themselves to be fully convinced of the veracity of all the arguments. Self discovery of results is indeed a very effective way to learn but we rarely have the time needed to use only that method - everyone needs to benefit from past thinkers!

There is (more than) enough here for a one semester course. Consequently I regret that there is nothing in the text on three dimensional geometry, nor is there anything on non-Euclidean geometries. To include these topics one would have to skimp on the Euclidean geometry of the plane. I believe that it is better to get a thorough grounding on the ideas and methods of Euclidean geometry in the plane before embarking on these other topics. These topics are indeed available in Geometry II (but we rarely get to teach it).

The notes are divided into eight thematic sections.

1 Axioms and models
2 Congruence, similarity, and Pythagoras
3 Constructions: algebraic and natural
4 Elementary geometry of the circle
5 Elementary area considerations
6 Concurrences for the triangle
7 Pole-polar duality and inversion
8 A few gems from here and there
Appendix: some axiom schemes for Euclidean geometry
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John Duncan, 2000
1 AXIOMS AND MODELS

For two millenia Euclid’s Elements held sway in the teaching of plane geometry. This dominance is understandable when we consider the Greeks’ monumental intellectual achievement of starting with problems of “measuring the earth” and ending up with an abstract formal deductive system. Nor should we think less of Euclid’s effort in the light of the inadequacies of his actual axiomatic system - these very inadequacies prompted much geometrical research including the formulation of several non-Euclidean geometries. Only within the last hundred years have we had (several) adequate axiomatic systems for Euclidean geometry. We do not intend here to study these axiomatics in detail; it is a worthy intellectual task but we personally find it somewhat dry. We want to do lots of geometry rather than wrangle over the foundations, but we have included a brief appendix on axiom systems of Euclidean geometry for your own reading.

Where then do we begin? We begin with a concrete model for the Euclidean plane. Instead of regarding points and lines as undefined objects that obey various rules, we shall say precisely what we mean by them. Of course our model will satisfy any of the equivalent axiom schemes for Euclidean geometry even though we shall not check this. For the source of our model we must return to the 17th century France of Descartes and Fermat who converted geometry to arithmetic (or algebra if you prefer). They continued to draw the standard geometrical figures (as we shall do) even though they used algebraic equations to represent lines, circles, etc.

The Euclidean plane is the set \( \mathbb{R}^2 \) of all ordered pairs \((x, y)\) where \(x, y \in \mathbb{R}\), the set of all real numbers. [It is vital to later arguments that we have the whole real number system available and not just, say, the rationals.] A point in the plane is an ordered pair \((x, y)\). A line in the plane is a subset of the form \(\ell = \{(x, y) : ax + by - c = 0\}\), where \(a, b, c \in \mathbb{R}\) and \((a, b) \neq (0, 0)\). For any \(k \neq 0\) we also have \(\ell = \{(x, y) : kax + kby - kc = 0\}\). We say that triples \((a, b, c)\) and \((p, q, r)\) are equivalent, written \((a, b, c) \sim (p, q, r)\) if there is some non-zero \(k\) with \(p = ka, q = kb, r = kc\). We write \([a, b, c]\) for the equivalence class of all triples \((p, q, r)\) that are equivalent to \((a, b, c)\). Thus there is a one-one correspondence between lines and equivalence classes \([a, b, c]\) with \(a, b, c \in \mathbb{R}\) and \(a, b\) not both-zero. It is sometimes helpful to regard \([a, b, c]\) as line coordinates. If the origin, i.e. \((0, 0)\), does not lie on the line then we have \(c \neq 0\) and so the line may be regarded as having equivalence class \([\alpha, \beta, 1]\) and thence may be identified with the point \((\alpha, \beta)\) in the plane. This sets up a
one-one correspondence between the points of $\mathbb{R}^2$ (except $(0,0)$) and the lines that omit the origin; this is called a point-line duality. In Euclidean geometry we do not have a complete duality between all the points and all the lines - that complete duality, which is very useful, is available in projective geometry. Notice how the algebraic viewpoint led us naturally to this duality; work out for yourself the actual geometrical correspondence between the “point” $(a,b)$ and the “line” $(a,b)$. [It will emerge in a later chapter!]

Facts about points and lines that we regard as intuitively obvious (alternatively that appear in an axiom system) can be checked by algebra. For example, given any two distinct points there is a unique line that contains both points - prove it! The dual statement: given any two distinct lines there is a unique point that both lines contain - is false in Euclidean geometry because of the presence of parallel lines. We say that lines $[a,b,c]$ and $[\alpha,\beta,\gamma]$ are parallel if there is $k \neq 0$ with $\alpha = ka, \beta = kb$. The correct dual statement in Euclidean geometry is: given any two distinct non-parallel lines there is a unique point that both lines contain. When two lines are parallel the two equations that represent them are algebraically inconsistent; in geometrical language, parallel lines never meet.

We need two more fundamental concepts; distance and angle. The distance between points $(a,b)$ and $(p,q)$ is the square root (non-negative, by fiat!) of $(p-a)^2 + (q-b)^2$. For pure (=synthetic) arguments we prefer to denote points by a single letter, say, $A = (a,b), P = (p,q), Z = (x,y)$. We write $AP$ for the distance between $A$ and $P$. Verify by algebra the vital triangle inequality that $AP \leq AZ + ZP$. The formula for distance is of course directly motivated by the classical theorem of Pythagoras; we shall discuss this later. The dual concept to distance is the angle between non-parallel lines. This is much more difficult to define and we need to detour first.

Note that distance immediately allows us to define a circle in the plane as a set of the form $\{P : AP = r\}$ for some radius $r > 0$; and we can represent it by an equation $(x-a)^2 + (y-b)^2 = r^2$. Sometimes it is convenient to consider a point as a degenerate circle with center $(a,b)$ and radius 0; sometimes it is convenient to consider a line as a degenerate circle with center at infinity. The collection of all circles is parametrized by three real numbers $a, b, r$ with $r > 0$ (or $r \geq 0$). Of course we do our geometry of lines and circles by drawing the “usual” pictures for them, as shown overleaf, and the more delicate aspects of the subject are infeasible for most of us without these pictures. In the diagram overleaf the line meets the circle at the points $A$ and $B$. There are two ways to go from $A$ to $B$ along the
circle, one shorter and the other longer. But if $A$ and $B$ were diametrically opposite, how could we distinguish the two routes? The intuitive response (the only one we shall give) is to use orientation; we can travel from $A$ to $B$ in the counterclockwise (or positive) direction, or we can travel from $A$ to $B$ in the clockwise (or negative) direction. Actually, when we talked above about the shorter route, how do we measure the length of an arc? The formal answer is to take the least upper bound of all finite approximations; a finite approximation being formed by marking off successive points from $A$ to $B$ and adding the lengths of all these straight line segments (for this definition it is essential that we have the real number system underlying our geometry). The informal answer is to lay a piece of thread along the arc of the circle and then measure the straightened thread on a ruler!

Even for a line segment $AB$ we may wish to distinguish between the distance “from $A$ to $B$” and “from $B$ to $A$” by calling one “plus” and the other “minus”. That distinction was elaborated by Felix Klein into a fascinating geometry text that he wrote for school teachers. Another fundamental question for some axiom schemes is what we mean by $P$ being between $A$ and $B$. Let $A = (a, b), B = (p, q)$. Then $P = (x, y)$ is between $A$ and $B$ if there is some $t$ with $0 \leq t \leq 1$ such that $x = ta + (1 - t)p, y = tb + (1 - t)q$. Of course $P$ is on the unique line determined by $A$ and $B$! We say that $P$ divides $AB$ in the ratio of $1 - t$ to $t$ and we write $AP : PB = (1 - t) : t$. For $t = 1/2$, $P$ is the mid-point of $AB$. The associated ray (or semi-infinite line) from $A$ through $B$ is the set of all points $P$, as above, with $t \geq 0$.

We may define angles as between lines or rays. (It comes to the same thing though it takes some work to see why!) We shall take the easier case of rays. Consider rays from $A$ through $B$ and from $A$ through $C$. Draw the circle center $A$ and radius one to intersect the rays at $P, Q$ respectively. The (directed) angle from $AB$ to $AC$ is the length of the arc of the circle measured from $P$ to $Q$ in the positive sense. So the angle from $AB$ all the way round
to $AB$ again is the length of the circumference of the circle - which we define as $2\pi$. We shall take as obvious (by symmetry) that a straight line $BAC$ has angle $\pi$. An angle of $\pi/2$ is called a right angle. We may also measure angles in degrees, instead of radians, by the equation:

$$2\pi \text{ radians} = 360 \text{ degrees}.$$ 

We denote an angle by $\angle BAC$.

Our last fundamental concept (for the Euclidean plane) is that of area. If you must, write out for yourself a definition of a rectangle. The area of a rectangle is the product of its width and height. (Yes, we are passing over the elementary discussion that opposite sides have the same length.) For a more general “region” we take the least upper bound of all finite approximations by rectangles; that is, we take a finite pairwise disjoint union of rectangles and add the individual areas. Just as for length, it is vital here that our model is based on the real number system. We shall take as obvious that the area of the disjoint union of two regions is the sum of the areas of the regions.

What then is Euclidean geometry? It is the study of properties of figures that are invariant with respect to where the figure is located in the plane. Thus if we move a triangle by sliding it, flipping it or rotating it, we do not change its geometrical properties. The characteristic property of such movements is that the distance between points is unchanged when we move to the new points; in technical language the movement is an isometry. We shall not prove it, but every isometry is a composition of translations, rotations and reflections. (Recall that we can reflect in a point or in a line.)

There are two other important algebraic viewpoints on the Euclidean plane. We can just as well represent a point by a complex number $z$ by associating $x+iy$ with the ordered pair $(x, y)$. We shall see later how complex
conjugation and complex multiplication provide wonderful tools for delicious proofs of many theorems. We can also associate the ordered pair \((x, y)\) with the vector \(x \hat{i} + y \hat{j}\). We shall see how vector algebra can also provide delicious proofs of many theorems. We shall also feel free to call upon the resources of trigonometry - which will provide yet more delicious proofs.

So as far as proofs as concerned we shall use anything we can lay our hands on! Wherever possible we prefer pure proofs, not because they are usually shorter but because they usually afford deeper insight (literally!) as to why something is true. In an entirely algebraic proof we may go through a long series of calculations and the final assertion may follow after some tortuous algebraic or trigonometric identities - later someone may achieve the algebra by a less tortuous route, but we are still left feeling ignorant of a geometrical reason for the result. In some cases the algebraic computations may be too horrendous to fight through and we may then be delighted to have a pure proof available. On the other hand we need to acknowledge that even pure proofs regularly use at least a modicum of elementary algebraic computation (e.g. factorizing a difference of squares). The chapters are designated by geometrical themes, but we have been strongly tempted to rewrite the chapters one for each of the many geometrical tools for proving results. But that approach may be better reserved for a second geometry course once we have a good picture of the central themes of Euclidean geometry.

This concludes our discussion of the foundations of Euclidean geometry in the plane. Time considerations prevent us from including higher dimensional geometry in this course. Our second course in geometry will include some of that as well as several other geometries - affine, inversive, projective,... The beginnings of these other geometries are adumbrated throughout this course and occasionally we shall give a brief indication of the good things to come! A brief synopsis of axiom schemes for Euclidean geometry is given in the Appendix.

FROM NOW ON, we shall freely use all the usual intuitive pictures for points, lines, circles, etc. If doubts arise about an obvious pictorial geometric truth, we can resort to algebraic verification on the basis of our algebraic descriptions. We shall not go out of our way to do this because it can be wearisome.
Two non-parallel lines intersect in one point. Three lines, no two of which are parallel, intersect in three points, say $A, B, C$. Triangle $ABC$ is made up of the line segments $AB, BC, CA$. The points $A, B, C$ are called the vertices. There is an internal angle at each vertex (hence the name triangle). The triangle is the simplest, and perhaps the most marvelous of all the figures in the plane; for an interesting discussion of human fascination with the triangle, see the article by Philip Davis in the American Mathematical Monthly, March, 1995. [As a side exercise, consider the possible figures we get from four lines, no two of which are parallel! What if we allow parallels? What about five or more points?]

Two triangles are congruent if we can move one exactly on top of another by an isometry; more precisely triangle $ABC$ is congruent to triangle $PQR$ if there is an isometry that takes $A$ to $P$, $B$ to $Q$, and $C$ to $R$. Of course, congruent triangles have their corresponding sides and angles equal. What partial information about sides and angles is sufficient to force two triangles to be congruent? There are four well known tests for congruence and one less well known test. In some school courses, some (or all) of the rules of congruence are taken as axioms; in other courses one proves them by “moving and superimposing” triangles. We shall take another approach recently suggested by Leonard Gillman (American Mathematical Monthly, 1994) The question may be reformulated as follows: what partial information about a triangle uniquely specifies the triangle (up to an isometric movement)? This viewpoint makes all the congruence rules pictorially obvious. We take the rules one at a time with their mnemonics and their construction pictures.

**SSS** Three side lengths uniquely determine a triangle.

Mark off one length as the base; draw arcs of circles for the other lengths as shown and we get unique point of intersection above the base (of course there is a congruent triangle reflected below the base). Note in passing that not any three numbers $a, b, c$ can be the side lengths of a triangle. We need
$a < b + c, b < c + a, c < a + b$. [If ever equality occurs then the triangle degenerates into a line segment.] More succinctly, we can assume without loss of generality that $a \leq b \leq c$ (why?) and then we need only that $c < a + b$ (prove this assertion!).

**SAS** Two side lengths and the included angle uniquely determine a triangle.

![Triangle with sides labeled](image)

The picture needs no comment!

**ASA** One side and the corresponding angles uniquely determine a triangle.

![Triangle with angles labeled](image)

Again, no comment.

**⊥HS** The hypotenuse and a side uniquely determine a right angled triangle.

![Right angled triangle](image)

Draw the side and right angle as shown and then draw an arc with radius the length of the hypotenuse. By the distance formula (or by Pythagoras, if you prefer) the hypotenuse is the longest side of a right angled triangle. So this congruence test is just a special case of the final lesser known one.

**SsA** Two sides and the angle opposite the larger side uniquely determine a triangle.
The construction is essentially as in the above case. We can easily see that two sides and the angle opposite the smaller side do not uniquely determine a triangle - there are two such triangles possible.

At this point, students may wish to go back and review some elementary exercises in congruence from school geometry. For example, prove that a quadrilateral has its opposite sides parallel if and only if it has its opposite sides equal in length. Prove also that a quadrilateral with two opposite sides parallel and equal in length is a parallelogram, etc.

It is very helpful here to recall the Sine and Cosine Rules (we shall discuss proofs of them later). These are fundamental to the numerical determination of triangles from partial data (as in surveying); they illumine the above congruence rules and they are extremely powerful tools for solving many problems in geometry. We are given a triangle with angles $A, B, C$ and opposite side lengths $a, b, c$.

**SINE RULE:** \[ a/\sin(A) = b/\sin(B) = c/\sin(C) \]

**COSINE RULE:** \[ a^2 = b^2 + c^2 - 2bc \cos(A), \quad b^2 = c^2 + a^2 - 2ca \cos(B), \quad c^2 = a^2 + b^2 - 2ab \cos(C) \]

How do these relate to the congruence rules? In $SSS$ we know $a, b, c$ and the Cosine Rule gives the cosine of each angle and hence each angle uniquely. In $SAS$ we know, say $b, c, A$. The Cosine Rule gives $a$ and thence $B, C$. In $ASA$ we know, say $a, B, C$. The Angle-Sum Theorem below gives us $C$, and then the Sine Rule gives $b,c$. In $\perp HS$ we have a plethora of possibilities. In $SsA$ we know, say $a, b, A$. The Sine Rule then gives us $\sin(B)$; but this gives two possible (supplementary) values for $B$. The additional information in $SsA$ forces $B$ to be acute and hence determines it uniquely; $c$ and $C$ follow. There are many elementary geometrical problems that can be solved by use
of the Sine Rule and/or the Cosine Rule. For example, try the Sine Rule for the following. Let the bisector of angle $A$ meet $BC$ at $X$. Prove that $BX : XC = AB : AC$. And conversely, this equality of ratios forces $AX$ to be the angle bisector at $A$.

**ANGLE-SUM THEOREM** *The sum of the internal angles of a triangle is $\pi$.*

Proof. Let the triangle be $ABC$. From the coordinate geometry viewpoint we can clearly draw the line through $A$ parallel to $BC$. Alternate angles are equal (from translation and reflection) and so the result is obvious from the diagram below. In the second diagram we have added pairs of parallel lines: $AB, A'C'$; and $AC, A'C'$. Check that triangles $ABC, CAA', A'CC'$ are congruent to each other, and deduce that any triangle will tile the plane. In particular, we get six copies of triangle $ABC$ making a complete revolution about $C$. An immediate Corollary is the external angle sum theorem; the external angle at any vertex of a triangle is the sum of the two opposite internal angles. Problem 45 gives one analogous internal angle sum theorem for “circular” triangles.

**PONS ASINORUM** *In triangle $ABC$ we have $AB = AC$ if and only if $\angle ABC = \angle ACB$.*

Proof 1. Let $AB = AC$. Triangles $ABC$ and $ACB$ are congruent by $SSS$. Let $\angle ABC = \angle ACB$. Triangles $ABC$ and $ACB$ are congruent by $ASA$.

Proof 2. Let $AB = AC$. If $M$ is the mid-point of $BC$ then triangles $ABM$ and $ACM$ are congruent by $SSS$. OR, if $P$ is the foot of the perpendicular
from $A$ to $BC$, then triangles $ABP$ and $ACP$ are congruent by $\perp HS$. Let $\angle ABC = \angle ACB$. If $P$ is the foot of the perpendicular from $A$ to $BC$, then triangles $ABP$ and $ACP$ have two angles equal and hence three angles equal by the angle-sum theorem; so they are congruent by $ASA$.

Proof 3. Let $P$ be as above. Then $\sin(B) = AP/AB$, and $\sin(C) = AP/AC$. The result follows. [Well - this really needs some more details about similar triangles - see later.]

The picture in Proof 2 looks like a simple suspension bridge and this proposition was the first stumbling block for weak geometry students - hence the mnemonic “bridge of asses”.

We pause for a long time favorite problem on a special isosceles triangle. In triangle $ABC$, we have $AB = AC$ and $\angle BAC = 20^\circ$; $P$ lies on $AC$ so that $AP = BC$. We have to find $\angle ABP$, say $\theta$. By Pons Asinorum and the Angle Sum Theorem we get $\angle BCA = 80^\circ$. The Sine Rule in triangle $ABP$ gives $BP/\sin 20 = AP/\sin \theta$. The Sine Rule in triangle $BPC$ gives $BP/\sin 80 = BC/\sin(\theta + 20)$. Since $AP = BC$, we get $\sin 20 \sin(\theta + 20) = \sin 80 \sin \theta$. But $\sin 80 = \cos 10$. The double angle formula for sine reduces this to $2 \sin 10 \sin(\theta + 20) = \sin \theta$. Our geometrical problem has unique solution, and since $\sin 30 = 1/2$, we see that the solution is given by $\theta = 10$. There is an elegant pure proof that begins by drawing triangle $APQ$ congruent to triangle $BCA$ (with $Q$ on the same side of $AC$ as $B$).

Formulate for yourself some congruence theorems for quadrilaterals. Perhaps you should stick to the convex case (each diagonal is “inside” the quadrilateral). Hinging four rods together should give you adequate insight! From the mechanical viewpoint, a triangle is rigid while a quadrilateral is not - just look at some old bridges! After the wonders of the triangle, quadrilaterals are full of surprises.

Consider the following congruence situation for a quadrilateral. We are given $AB = a$ and we specify $C$ by $\angle BAC = \alpha$ and $\angle ABC = \beta$. We also specify $D$ (on the same side of $AB$ as $C$) by $\angle BAD = \gamma$ and $\angle ABD = \delta$. Clearly this uniquely determines quadrilateral $ABCD$. Now we wish to calculate all the angles in the picture. Let the diagonals $AC$ and $BD$ meet
at $E$. Then $\angle AEB = \angle CED = \pi - \alpha - \delta$, $\angle AED = \angle BEC = \alpha + \delta$, $\angle ACB = \pi - \alpha - \beta$ and $\angle ADB = \pi - \gamma - \delta$. It remains to calculate $\angle ACD$ and $\angle BDC$. Up till now all the angles have been given by linear combinations of the initial data. This is not the case for the remaining two angles. The angle sum theorem will give us the sum of these two unknown angles, but not the actual angles themselves. Of course we can use the Sine Rule to calculate $AC, AD, \angle ACD$ in succession, but the final angle involves solving a messy trigonometric equation.

Triangles $ABC$ and $PQR$ are similar if $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ (note the shorthand!) and if there is some $k > 0$ such that $PQ = kAB, QR = kBC, RP = kCA$; thus one triangle is just a “blown-up” version of the other. As with congruence of triangles we ask what partial information is sufficient to make two triangles similar. Consider first the situation in which two triangles agree in one of their angles. By an isometric movement we may suppose without loss that one triangle is inside the other as in the diagram below. The central idea behind all theorems on similarity is that the ratios $BP : BA$ and $BQ : BC$ are equal if and only if $PQ \parallel AC$ (i.e. $PQ$ is parallel to $AC$). You may easily check this assertion for yourself by a coordinate geometry argument with $B$ as the origin, but this is a good point to begin to show the power of vector arguments. Let arrows $BQ, BP$ represent vectors $\vec{u}, \vec{v}$ respectively. Hence, arrows $BC, BA$ represent vectors $\lambda \vec{u}, \mu \vec{v}$ respectively. Then, arrows $PQ, AC$ represent vectors $\vec{u} - \vec{v}, \lambda \vec{u} - \mu \vec{v}$
respectively. If the ratios are equal, then \( \lambda = \mu \), \( \lambda \vec{u} - \mu \vec{v} = \lambda(\vec{u} - \vec{v}) \), and so \( PQ \parallel AC \). Conversely, if \( PQ \parallel AC \), then \( \lambda \vec{u} - \mu \vec{v} = \nu(\vec{u} - \vec{v}) \). This gives \( (\lambda - \nu)\vec{u} = (\mu - \nu)\vec{v} \). Since \( P \) does not lie on \( BQ \) this can happen if and only if \( \lambda - \mu = 0 = \mu - \nu \). Hence the ratios are equal, as required.

Now we can apply this fundamental fact to derive criteria for similarity. If all three angles agree (criterion \( AAA \)), then \( PQ \parallel AC \) and all three ratios agree by the above reasoning. Equally, if the two ratios agree on the sides along the common angle \( B \) (criterion \( SAS \)), then \( PQ \parallel AC \) and hence all angles agree and we have similarity.

If two triangles have their corresponding sides in the same ratio (criterion \( SSS \)), then the Cosine Rule shows that their corresponding angles are equal, and again we have similarity. Conceptually it is nice to think of similar triangles as a generalization of congruent triangles; in practice we most commonly use \( AAA \), and occasionally \( SAS \). (Similarity for quadrilaterals is again full of surprises.) As an application we shall prove the Theorem of Pythagoras using similarity.

**THEOREM OF PYTHAGORAS** The square of the hypotenuse of a right angled triangle is the sum of the squares on the other two sides.

Proof. Let \( M \) be on \( AB \) with \( CM \) perpendicular to \( AB \). We easily see that triangles \( AMC, CMB, ACB \) are all similar. It follows that \( AM : AC = AC : AB, BM : BC = BC : BA \) and so \( AC^2 + BC^2 = AM \cdot AB + MB \cdot AB = AB^2 \).

There are of course many proofs of this theorem (including one by President Garfield). The above is my favorite proof, but many prefer the following delicious one. There are four congruent triangles in the diagram; two of these fit together to give a rectangle sides \( a, b \). By the additive property of the area function we get \( (a + b)^2 = c^2 + 2ab \), and so \( a^2 + b^2 = c^2 \). However, we have omitted the initial construction of the diagram. Start with the square of side \( c \) and then place the four \((a, b, c)\) triangles around it. But you have to prove that the outer figure is actually a square! This illustrates a well known truth that \textit{proofs without words} can be quite long when all the details are written out!
We also have the converse of the Theorem of Pythagoras: if $AB^2 = AC^2 + CB^2$ then $\angle ACB$ is a right angle. Let the perpendicular from $A$ meet $BC$ at $C^*$. Apply Pythagoras to triangles $AC^*C$ and $AC^*B$ and derive a contradiction unless $C = C^*$.

Some applications of Pythagoras may need some preliminary work. Consider, for example, the TV screen problem (\#48). Let the arcs of the TV screen intersect to give the rectangle (why?) $ABCD$. An isosceles triangle with an angle of $60^\circ$ is equilateral, and so the height $BC$ is $15cm$ and the width is $15\sqrt{3}cm$. The defining property of the arcs means that the arcs extend the same distance beyond the rectangle $ABCD$. Let the height extension be $y cm$. Then the frame has height $(15 + 2y)cm$. But this radius is also the hypotenuse of a right angled triangle with sides $15+y$ and $(15\sqrt{3})/2$. This gives an equation to solve for $y$ and hence we get the height of the frame. A similar argument gives the width of the frame and then finally Pythagoras gives the diagonal of the frame. We leave the details to you.

We recall also that similarity is a crucial aspect of trigonometry. Let us define $\cos \theta$ and $\sin \theta$ to be the coordinates of the point $P$ on the unit circle that makes angle $\theta$ with the positive $x$-axis. By similar triangles we get the usual formula $\cos \theta = \text{adjacent:hypotenuse}$ and also $\sin \theta = \text{opposite:hypotenuse}$. 

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site: hypotenuse. So \( \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \), and so on. This last formula gives an unambiguous definition for the slope of a line - except that we have the annoying special case of a vertical line whose slope is not defined (unless we say it has infinite slope). One of the uses of slope is to specify the “direction” of a line away from a given point on that line; in fact it is preferable to do this by a vector \( a\mathbf{i} + b\mathbf{j} \) and then a vertical line has direction specified by \( \mathbf{j} \).

We take the opportunity here to recall some other simple ideas about vectors in the plane. Three line segments that represent vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) constitute the sides of a triangle if and only if \( \mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0} \). The scalar product \( \mathbf{u} \cdot \mathbf{v} \) of two vectors \( \mathbf{u}, \mathbf{v} \) can be defined either as \( |u||v| \cos(\theta) \) where \( \theta \) is the “angle between the vectors”, or by the algebraic formula \( ac + bd \), where \( \mathbf{u} = a\mathbf{i} + b\mathbf{j}, \mathbf{v} = c\mathbf{i} + d\mathbf{j} \). (It is a nice application of the Law of Cosines to show that the algebraic definition has the stated geometrical property.) In particular, the associated line segments are at right angles if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \). With these simple tools, we can often reduce a geometrical problem to some algebraic manipulations with vectors. Sometimes it is convenient to replace the “basis” vectors \( \mathbf{i}, \mathbf{j} \) with two other basis vectors (i.e. any two vectors that are not multiples of each other). It is not a priori obvious which geometrical problems should be tackled by vectors; but if you are making no headway by pure methods you should always try using vectors!

As an illustration, we consider the following problem that comes out easily by vectors. We are given a quadrilateral \( ABCD \) and points \( P, Q \) such that \( DACP \) and \( ABDQ \) are parallelograms. Show that \( AC = BD \) if and only if \( BP \) is perpendicular to \( CQ \). Label \( AB, BC, CD \) by the vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) as in the diagram and then label the other sides by the rules of vectors.

PSfrag replacements
Then $AC = BD$ is equivalent to $(\vec{u} + \vec{v}).(\vec{u} + \vec{v}) = (\vec{v} + \vec{w}).(\vec{v} + \vec{w})$, or $\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} = \vec{w} \cdot \vec{w} + 2\vec{w} \cdot \vec{v}$, or $(\vec{w} - \vec{u}).(\vec{u} + \vec{w} + 2\vec{v}) = \vec{0}$. But the last equation simply says that $BP$ is perpendicular to $CQ$.

We have to work somewhat harder for pure proofs. Suppose that $AC = BD$. Then $AQ = BD = AC$ and $DP = AC = DB$ and so we get two isosceles triangles. By parallel lines, angles $QAC$ and $PDB$ add to 180, and hence angles $AQC$ and $PBD$ add to 90. By two rotations we deduce that $QC \perp BP$. Conversely, suppose that $QC \perp BP$. Then angles $AQC$ and $PBD$ add to 90, as do angles $QCA$ and $BPD$. Suppose, if possible, that $BD > AC$. Then $AQ > AC$ and so $\angle QCA > \angle AQC$ [this follows from the Sine Rule, since the angles are acute]. Similarly we get $BD > DP$ and so $\angle BPD > \angle PBD$. Addition gives the contradiction that $90 > 90$. So we cannot have $BD > AC$. Similar reasoning shows that we cannot have $BD < AC$. Hence $BD = AC$. But, of course, we prefer the vector argument for this problem.
3 CONSTRUCTIONS: ALGEBRAIC AND NATURAL

The Greeks were interested not just in properties of figures but also in how to construct aspects of those figures. For example, given two circles with neither inside the other, it is intuitively obvious that various common tangent lines exist - but how can we draw such common tangent lines? One obvious response is to simultaneously move a pencil round the first circle with a straightedge attached that is tangent to the circle at that point - then stop when the straightedge is tangent to the second circle. What kind of instrument do we need to do this? Not one available to either the Greeks or to us! The Greeks allowed only a straightedge (not a ruler with markings on it) and a compass (sometimes called a *pair of compasses* since the instrument has two legs). The Greeks handicapped themselves even more by allowing only a *collapsing compass*; once you draw a circle and remove the instrument from the paper the compass immediately collapses so that you could not copy directly this circle to another with a different center but the same radius. In fact it is possible to do this circle copying with straightedge and collapsing compass (as we show later); indeed any construction that is possible with straightedge and modern compass is possible with straightedge and collapsing compass (with much more effort, of course). Apart from the special construction mentioned above, we shall confine our attention to the modern compass. (Personally I regard the collapsing compass viewpoint as silly - how did the Greeks know that their collapsing compass did not collapse in the middle of drawing a circle?) Modern technology has produced other instruments - Miras, computer drawing packages, etc. We shall ignore these (for good mathematical reasons).

There are plenty of other variants for amusement. Suppose the compass has a fixed radius. Suppose we replace the compass by a circular disk of fixed radius; given two points suitably close together we can draw either of the two circles through these points with the radius of our disk. See Problem 177 for a fascinating example. When we do practical constructions we use a finite sheet of paper and we sometimes run into the following kind of difficulty. We have two lines on the paper that intersect at a point $P$ off the paper, and we want to join $P$ to another point $Q$ on the paper; how can we construct within the paper a portion of the line $PQ$? We leave you to devise other variants.

What then can we do with straightedge and compass? The duller approach is to adopt the viewpoint of coordinate geometry. Start with a given
figure of points, lines and circles (it might just be a triangle) and specify some target point to be achieved by intersecting lines, lines and circles, circles and circles and so on. The initial figure is specified in coordinates by a finite number of parameters; the target point has parameters that are functions of these parameters. What kind of functions can appear? A little coordinate geometry assures you that we can form new numbers only by adding, subtracting, multiplying, dividing and taking square roots of the numbers we have at any stage. If we can achieve these algebraic operations by geometric construction then we can “construct” our target point and the goal is achieved. It is easy to add and subtract numbers; how do we achieve the other three starting with a unit 1 and lengths a,b? Study the three diagrams below!

To achieve these constructions by straightedge and compass we need to be able to do the following simple constructions. *Through a given point draw the line parallel to another given line; bisect a given line segment; given a point on a line erect the perpendicular to the line through that point.* We leave these as refreshers on school geometry. This appears to finish the subject of constructions, but there are two hitches. Doing the actual symbolic coordinate geometry computations may be too formidable (this is indeed the case more often than not!); even when we can do them, the associated construction (built out of the above three simple ones) may have a huge number of steps. So we really would like to have natural, practical, efficient constructions. Nonetheless, the algebraic viewpoint has led to some startling conclusions. The ancients tried to *duplicate the cube* (construct a cube whose volume is twice that of a given cube) by straightedge and compass constructions; this amounts algebraically to constructing the cube root of 2. By substantial use of the theory of equations this can be shown to be *impossible* (consult any good book on algebraic number theory). Similarly, it is *impossible*, by straightedge and compass, to *square the circle* (construct a square whose area is that of a circle of radius one) or *trisect an arbitrary angle* (we can’t even do 60 degrees!). Despite these impossibilities, mathematical cranks continue to come forward with “straightedge and compass
constructions” for these famous problems (consult the two recent books by Underwood Dudley: *Mathematical Cranks* and *A Budget of Trisections*).

Most elementary constructions are done in school, so let’s warm up on the problem mentioned above. How do we copy a circle using only straightedge and collapsing compass? Here is the construction - you have to justify it (hint: use congruent triangles). We are given a circle center $B$ and radius $r$, and a point $A$. Suppose $A$ is outside the circle (you deal with the other case). We can draw the circles center $A$ and $B$ with radius $AB$. Let them intersect at $C$ and $D$. Let $E$ be one point of intersection of the original circle and this circle center $A$. Now construct the circle with center $C$ and radius $CE$; let it meet the circle center $B$, radius $AB$ at $P$. Then $AP = r$ and so we are done.

Now recall for yourself the following constructions that are fundamental to the geometry of the triangle. *Given a line and a point $P$ off the line, construct the line through $P$ at right angles to the given line.* Construct the (internal) bisector of a given angle. *Given lengths $a, b$, divide a given line segment internally (and externally) in the ratio $a : b$.*

The Greeks had a great interest in the *golden mean* (or the golden ratio) given by $\mu = (1 + \sqrt{5})/2$. It satisfies the equation $\mu^2 = \mu + 1$ (or equivalently, $\mu = 1/\mu + 1$). A rectangle with sides $a, b$ ($a < b$) is *golden* if $a/b = 1/\mu$. If we cut off a square side $a$, the remaining rectangle is again golden (prove it!); the construction can be continued indefinitely and if we make the right choices at each stage we can produce a beautiful spiral with intriguing properties (see Problem 128). [The golden mean played an important role in paintings.] Of course we can construct the golden mean by the dreary method, but here is Euclid’s delightful construction related to the figure below.

Let $ABCD$ be a square of side $a$, and $M$ the midpoint of $AB$. Draw the circle center $M$ and radius $MC$ to meet $AB$ produced at $E$. The completed rectangle $AEFD$ is a golden rectangle. Euclid used this in his straightedge and compass construction of the regular pentagon (discussed below).
There are many interesting, fundamental constructions involving circles but we shall defer these until we have discussed some geometry of the circle. Some of these circle constructions are rather difficult!

It is a remarkable fact that all constructions that can be done by straight-edge and compass can actually be done by compass only. This is the famous Mohr-Mascheroni Theorem obtained independently by Mohr in 1672 and Mascheroni in 1797. It is sufficient to show that just a few fundamental constructions can be done using compass only - the proof is tedious rather than difficult - bright students regularly discover it for themselves if it is not included in their own course. A very elementary proof has recently been published in the American Mathematical Monthly by Hungerbuhler (October 1994).

Finally, let us consider the astonishing regular pentagon/pentagram. Justify all the named angles in the diagram below and conclude that $\theta = \pi/5 = 36^\circ$. Let $d$ be the diagonal length, $\ell$ the side length and $y, x$ be as shown. Use similar triangles to show that $d : \ell = \ell : y = y : x$. Since $\ell = x + y$, deduce that each ratio is the golden ratio $\mu$ with $\mu = 1 + 1/\mu$. The Sine Rule gives $\mu = \sin(2\theta)/\sin \theta$ and hence we get $\cos \theta = \mu/2$. You now have enough information to devise a straightedge and compass construction of the regular pentagon.
4 ELEMENTARY GEOMETRY OF THE CIRCLE

Consider the following diagram; it is just the *Pons Asinorum* diagram with a circle drawn (and so perforce the triangle is isosceles). The line from $O$ to $M$, the mid-point of $AB$ is clearly perpendicular to $AB$; on the other hand, the perpendicular bisector of $AB$ passes through the center $O$ of the circle. In particular there are infinitely many circles passing through $A$ and $B$. If we mark any other point $P$ on the circle then the perpendicular bisector of $BP$ also passes through the center $O$. This gives us a Euclidean construction to find the center of any given circle; it also shows that any three non-collinear points uniquely determine a circle through them.

DOUBLE ANGLE THEOREM

As in the diagram below, the angle subtended at the center of the circle is twice the angle at the circumference. In particular, all (acute) angles on the circumference subtended by $AB$ are equal.

![Diagram](image1.png)

Proof. We apply the external angle sum theorem twice with isosceles triangles; in the first picture we add the angles, in the second we subtract.

As a special case we note that any angle subtended by a diameter is a right angle. We leave you to do the converse: given segment $AB$ the locus of points $P$ with $\angle APB$ a right angle is a circle with diameter $AB$. More generally, the locus of points with $\angle APB$ a fixed acute angle is the major arc of a circle.
with \( AB \) as a chord. What happens if \( P \) is on the minor arc? We have noted that any three non-collinear points uniquely determine a circle. Clearly it may be impossible to find a circle to pass through any four given points. We say that a quadrilateral \( ABCD \) is cyclic if there is a circle passing through

\[
\begin{align*}
A, \ B, \ C, \ D. \quad \text{It follows from the double angle theorem that opposite angles are} \\
\text{supplementary (i.e. add to 180 degrees); equivalently each external angle is} \\
\text{equal to the opposite internal angle. Prove the converse (by a contradiction} \\
\text{argument). Another equivalent formulation is that} \quad \angle A + \angle C = \angle B + \angle D. \\
\text{In similar vein, given a triangle} \ ABC, \ \text{the locus of points} \ P \ \text{so that} \quad \angle APB \\
\text{and} \ \angle ACB \ \text{are supplementary is an arc of a circle.} \\
\end{align*}
\]

For arbitrary curves we use calculus to define the tangent line at a point. For a circle we may use a simple intrinsic definition. Let \( P \) be a point on a circle with center \( O \). The tangent line at \( P \) is the line through \( P \) at right angles to \( OP \). It is immediate that the tangent line meets the circle at just one point (hence the algebraic definition); for if it meets the circle again at \( Q \) and if \( R \) is the mid-point of \( PQ \), then triangle \( OPR \) has two of its angles right angles! In the cyclic quadrilateral \( ABCD \) above, if we let \( C \\
\]

and \( D \) run together to \( T \), it is intuitively clear that the chord \( CD \) “ends
up as” the tangent line at T; so it is no surprise that the angle between the
tangent at T and TB is the angle at A. This is quite clear from the diagram
in one special case (which proves the general case - why?). Note also that
$PT^2 = PO^2 - OT^2 = PO^2 - OB^2 = PB.PA$. [Or, use the similar triangles
$PBT$ and $PTA$.] More generally, apply similar triangles in the following
diagrams to get $PA.PB = PC.PD$ in each case!

Consider now some classical construction problems for circles. Given a
circle center O and P outside it, how do we construct the two tangent lines
to the circle from P? Look at the picture one page back to see that we just
have to draw the circle on the diameter OP. Suppose now we are given two

circles as in the diagram. How do we construct all the common tangents?
Amongst the standard proofs, the nicest approach is to “collapse” one circle
to a point, as hinted in the diagram (use radii $R - r$ and $R + r$) — over to
you for the rest.

We say that two circles touch at P if they have the same tangent line at P
(sometimes we distinguish internal and external touching). Given two circles
we can find an infinite number of circles that touch these two circles (what is
the locus of centers of such circles?). Suppose we are given three circles. How
many circles touch each of these three? This is a famous classical problem
that we leave you to wrestle with.
The circle of Apollonius is the solution to a natural locus problem. Given fixed points $A, B$ and a fixed ratio $\lambda : \mu$, find the locus of all points $P$ so that $AP : PB = \lambda : \mu$. We can quickly see that the locus is a circle by coordinate methods. Take $A = (0, 0), B = (b, 0), P = (x, y)$. Then the locus is specified by $(\mu AP)^2 = (\lambda PB)^2$, or $\mu^2(x^2 + y^2) = \lambda^2[(x - b)^2 + y^2]$. Clearly this is a circle (when $\lambda = \mu$, we get the degenerate circle given by the perpendicular bisector of $AB$). Divide $AB$ internally and externally in the ratio $\lambda : \mu$ at $P, Q$ respectively. It is easy to see that the circle of Apollonius is just the circle with diameter $PQ$. [The locus of points $P$ so that the sum (difference) of $AP$, $BP$ is constant gives an ellipse (hyperbola) and hence is not within the purview of this course. The case when the product of the distances is constant does not even give a conic section.]

There is an interesting dual notion to the idea of a cyclic quadrilateral. We say that quadrilateral $ABCD$ is co-cyclic if the four sides of the quadrilateral are tangent lines to a circle inside the quadrilateral; alternatively we say that $ABCD$ circumscribes a circle. The duality is between a point on the circle and the tangent line at that point on the circle. Recall that $ABCD$ is cyclic if and only if $\angle A + \angle C = \angle B + \angle D$. In view of the duality between “angle” and “distance” we might guess that $ABCD$ is co-cyclic if and only if $AB + CD = BC + DA$. This is true! If $ABCD$ is co-cyclic, we have four pairs of common tangents from the vertices. Then $AB + CD$ is the sum of these four lengths - and so is $BC + DA$. For the converse draw the circle touching three of the sides. Mimic the proof of the converse result for cyclic quadrilaterals to derive a contradiction if the fourth side is not a tangent line also. [Actually the proof requires the knowledge that a circle can be drawn to touch three of the sides of $ABCD$; this theme will be discussed later.] Co-cyclic quadrilaterals have been strangely neglected in traditional geometry courses; we have included several nice exercises on them in the Problem List.
5 ELEMENTARY AREA CONSIDERATIONS

As noted earlier, a rectangle with sides $a, b$ has area $ab$. It follows (since area is additive) that a parallelogram base $a$, height $h$ has area $ah$ (just move a triangle across as in the diagram). It follows by symmetry that a triangle base $a$, height $h$ has area $\Delta = ah/2$. [Aside: Triangles with the same base have areas proportional to their heights; triangles with the same height have areas proportional to their bases. These simple facts turn out to be enormously powerful tools in many proofs.] By simple trigonometry we get the three formulas: $2\Delta = bc \sin A = ca \sin B = ab \sin C$. Divide throughout by $abc$ and we get the Sine Rule:

$$\frac{2\Delta}{abc} = \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$ 

For perhaps better insight into the Sine Rule consider the diagram shown to see that $a = 2R \sin A$ (etc) and so $4R\Delta = abc$. We might as well pause for the Cosine Rule. Apply Pythagoras twice in the diagram to get

$$a^2 = c^2 + x^2 - y^2 = c^2 + (x + y)^2 - 2xy - 2y^2 = c^2 + b^2 - 2bc \cos A$$

and two other formulas with the symbols suitably permuted. Check for yourself the case when angle $A$ is obtuse. Note that this calculates the angles
of a triangle given the side lengths. In the above formulas for the area of a triangle we can thus use trigonometric identities to get a formula for the area of a triangle (attributed to Heron) entirely in terms of the sides.

We have $4\Delta^2 = b^2c^2\sin^2 A = b^2c^2(1 - \cos^2 A)$. By the Cosine Rule we have $\cos A = (b^2 + c^2 - a^2)/2bc$. Substitution gives:

\[
16\Delta^2 = [(2bc)^2 - (b^2 + c^2 - a^2)^2] = a^2 - (b - c)^2[(b + c)^2 - a^2] = (a + b + c)(b + c - a)(c + a - b)(a + b - c).
\]

Let $s$ be the semiperimeter, $s = (a + b + c)/2$ and we get

\[
\Delta = \sqrt{s(s - a)(s - b)(s - c)}.
\]

Knowing the area of a triangle we can work out the area of any quadrilateral, or more generally, polygon, by decomposing into triangles. In particular, inscribe a regular $n$-gon inside a circle of radius $r$. We easily get the area of the $n$-gon as one half the perimeter times the altitude. “In the limit”, the perimeter becomes $2\pi r$ (there is a little similarity argument here we omitted earlier) and the altitude becomes $r$. So the area of a circle is $\pi r^2$. For an alternative viewpoint, calculate the area of the $n$-gon as $nr^2\sin(\pi/n)\cos(\pi/n)$ and let $n$ go to infinity.

Some of the results in the last section may be regarded as area theorems since they involve products of lengths; this explains the inclusion in this section of the following remarkable theorem.

THEOREM OF PTOLEMY  Let $ABCD$ be a convex quadrilateral. Then we have

\[
AC.BD \leq AB.CD + BC.DA
\]

with equality if and only if $ABCD$ is cyclic.
Proof. Name the lengths as in the diagram and construct triangle $AOB$ similar to triangle $ACD$, external to the quadrilateral. Deduce (by proportional sides and included angle) that triangles $OAC$ and $BAD$ are similar. It follows that $OB = ac/d, BC = bd/d, OC = xy/d$. The general inequality follows from the triangle inequality. Also, equality occurs if and only if $OBC$ is a straight line, if and only if angles $B$ and $D$ are supplementary, if and only if $ABCD$ is cyclic. What a gem!

Ptolemy’s Theorem for a cyclic quadrilateral can also be proved by trigonometric identities starting with the diagram below and using the angles subtended at the center of the circle by the quadrilateral. The Heron formula for the area of a triangle has a remarkable extension to the case of a cyclic quadrilateral, due to Brahmagupta:

$$K = \sqrt{[(s - a)(s - b)(s - c)(s - d)]}$$

where the sides are $a, b, c, d$ and $s$ is the semiperimeter, $s = (a + b + c + d)/2$. In the above diagram let $\theta$ be the angle between the diagonals $AC$ and $BD$.

Before we prove the Brahmagupta formula, we derive some formulas that are valid for any convex quadrilateral. In the above diagram, let $AC = m$ and $BD = n$. Apply the Cosine Rule to each of the triangles
To derive the following generalization of the Cosine Rule:
\[ a^2 - b^2 + c^2 - d^2 = 2mn \cos \theta. \]

Now use the \( "(1/2)bc \sin A" \) triangle formula to deduce Bretschneider’s formula for the area of any convex quadrilateral:
\[ 16K^2 = 4m^2n^2 - (a^2 - b^2 + c^2 - d^2)^2. \]

Finally use Ptolemy formula \( mn = ac + bd \) to get the Brahmagupta formula for a cyclic quadrilateral. After that warm up you may wish to go on to show that any convex quadrilateral has area formula:
\[ K^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2[(B + D)/2]. \]

Or, you may also like to show that a quadrilateral which is both cyclic and co-cyclic has the delicious area formula:
\[ K^2 = abcd. \]

Another famous area formula is associated with a polygon each of whose vertices has integer coordinates. We’ll suppose the polygon is convex and leave you to show that the formula works in general. The formula now appears in most school geometry texts, but perhaps is not proved in any of them. The proof is not trivial even for a triangle (and it is easy to reduce the general case to that of a triangle). There are lots of proofs available; here is our favorite. By a lattice polygon we mean a polygon all of whose vertices are lattice points, i.e. have integer coordinates. We remark that Pick’s Theorem is theoretically interesting, especially since there is no known analog in three dimensions. It is not of practical value; even for lattice polygons it is simpler to use the determinantal formula for the area that is derived from Green’s Theorem.

**PICK’S THEOREM** The area of a lattice polygon is \( i + (b/2) - 1 \) where \( i \) is the number of lattice points interior to the polygon and \( b \) is the number of lattice points on the boundary of the polygon.

Proof. Decompose the lattice polygon into lattice triangles and show that it is enough to verify the formula for a general lattice triangle. Such a triangle can be written as a difference of special triangles with their shorter legs on lattice lines) as in the diagram. Now prove that it is enough to verify the
formula for these special triangles. The area of such a triangle is half the area of the associated rectangle, so we are left to verify the area formula for a lattice rectangle - which is an easy exercise.

The Problem List has many problems on area. We now consider a few sample problems in area.

Let $ABC$ be any triangle and let $D, E, F$ be the trisection points of $BC, CA, AB$ respectively. Thus $BD : DC = CE : EA = AF : FB = 1 : 2$. Join $AD, BE, CF$ to form triangle $PQR$ as in the diagram below. Then $\Delta PQR = \frac{1}{7}\Delta ABC$. We show first that $BP : PQ : QE = 1 : 3 : 3$, and similarly for the other two lines inside the triangle. Triangles $ADB$ and $ADE$ have the same base and so their areas are in the ratio of their slant heights. Thus $BP : PE = \Delta ADB : \Delta ADE$. But triangles $ABD$ and $ABC$ have the same height. Hence, $\Delta ADB = (1/3)\Delta$. Analogously, $\Delta ADE = (2/3)\DeltaADC = (2/3)(2/3)\Delta$. It follows that $BP : PE = 3 : 4$. Use this method to show also that $BQ : QE = 6 : 1$. Now we get

$$\Delta PQR = \frac{3}{7}\Delta BER = \frac{3}{7}\frac{3}{2}\Delta BEA = \frac{3}{7}\frac{3}{2}\frac{2}{3}\Delta = \frac{1}{7}\Delta.$$  

Given a convex quadrilateral $ABCD$ can we find a point $O$ inside the quadrilateral so that the four triangles formed by joining $O$ to the vertices
all have the same area? We leave you to prove that such a point \( O \) exists if and only if at least one of the diagonals passes through the mid-point of the other diagonal. Now change the problem slightly. Let \( ABCD \) be a convex quadrilateral and let \( E, F, G, H \) be the mid-points of \( AB, BC, CD, DA \) respectively. Can we find a point \( O \) so that the lines \( OE, OF, OG, OH \) divide the quadrilateral into four smaller quadrilaterals each of the same area? Now we can always do it!

Let \( M, N \) be the mid-points of \( AC, BD \) as above. Complete the parallelogram as shown. Then \( O \) is the requisite point. Use similar triangles to show that \( \triangle EBF = (1/4)\triangle ABC \). Since \( ON \parallel AC \parallel EF \), we have \( \triangle EOF = \triangle ENF = (1/4)\triangle ADC \). Hence, quadrilateral \( OEBF \) is \( (1/4) \) of quadrilateral \( ABCD \), and similarly for each of the other three quadrilaterals.

Next, let us return to the astonishing regular pentagon/pentagram. There are five triangles in the diagram of decreasing area; show that the ratio of each to the next is the Golden Ratio \( \mu \)! Show that the ratio of the areas of the given pentagon and the inner pentagon is \( \mu^4 \). As further exercises, express the radius of the circumcircle in terms of \( \ell \) and \( \mu \); hence, or otherwise, find the area of the pentagon in terms of \( \ell \) and \( \mu \).
6 CONCURRENCES FOR THE TRIANGLE

Draw three lines in the plane at random. Virtually never will they be concurrent in some point. For the triangle there are lots of special line triplets that are always concurrent. We list together here the most important special lines:

PERPENDICULAR BISECTOR OF A SIDE - self-explanatory!
MEDIAN - a line from a vertex to the mid-point of the opposite side.
ALTITUDE - a line from a vertex that is perpendicular to the opposite side.
INTERNAL ANGLE BISECTOR - a line from a vertex that bisects the angle of the triangle at that vertex.

One theorem is tacit from our earlier discussion of the circumcircle of a triangle:

The perpendicular bisectors of the sides of a triangle are concurrent at the center of the circumcircle.

The concurrence of the medians of a triangle comes with some special information about the point of concurrence.

The medians of a triangle are concurrent at the centroid which divides each median in the ratio 2:1.

Proof 1. Let medians $BE, CF$ meet at $G$, and let $M, N$ be the mid-points of $BG, CG$, as in the diagram. $MN$ and $FE$ are each parallel to $BC$ and one half its length. So $MNEF$ is a parallelogram and so the diagonals bisect each other. So any two medians are trisected by their point of intersection. The result follows - but you should explain the details to yourself.
Proof 2. Let $BE, CF$ be as above and produce $AG$ to meet $BC$ at $D$. By elementary area considerations, $\Delta AFG = \Delta BFG, \Delta AFC = \Delta BFC$ and hence $\Delta AGC = \Delta BGC$. Similarly $\Delta BGA = \Delta BGC$. So each is one third of $\Delta ABC$ and again we have trisection of any two medians at the intersection point.

Proof 3. Let $A, B, C$ have coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Then $D$ has coordinates $((x_2 + x_3)/2, (y_2 + y_3)/2)$ and similarly for $E, F$. Now verify that the point $((x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3)$ divides each median in the ratio 1:2.

Proof 4. This is really just a variant of Proof 3, but it is convenient here to introduce a vector notation that we shall use later. Take the origin $O$ at the circumcenter of the triangle and let the directed line segments $OA, OB, OC$ represent vectors $\vec{u}, \vec{v}, \vec{w}$. Let $G$ be the point such that the directed line segment $OG$ represents the vector $(1/3)(\vec{u} + \vec{v} + \vec{w})$. A simple vector calculation verifies that $G$ lies on $AD$ with $AG = 2GD$, and similarly for the other two medians.

Notice (give two proofs) that $G$ is also the centroid of triangle $DEF$. In the diagram below, $FE$ is extended so that $FE = EK$. Thus $BDKE$ is a parallelogram and $DK = BE$. Show also that $FA$ is equal and parallel to $CK$, so that $AK = CF$. Thus the side lengths of triangle $ADK$ are the medians of triangle $ABC$. Show that the medians of triangle $ADK$ are $(3/4)$ the sides of $ABC$ and also that $\Delta ADK = (3/4)\Delta ABC$. If $p$ is the perimeter of triangle $ABC$, show that the sum of the medians of $ABC$ lies between $(3/4)p$ and $p$. [Hint: $AG + GC > AC$ and similarly for two other inequalities. Add to deduce that $AD + BE + CF > 3p/4$. Apply this inequality to triangle ADK to get the other inequality.]
At each vertex of a triangle we have an internal angle and an external angle; we can bisect each to get the internal bisector and the external bisector — of course these bisectors are at right angles to each other.

The internal bisectors of the angles of a triangle are concurrent at the center of the incircle.

Proof. Let the bisectors of angles $B$ and $C$ meet at $I$. Drop perpendiculars from $I$ to the sides as in the diagram. Triangles $BZI$ and $BXI$ are congruent and so $IZ = IX$. Similarly $IY = IX$. So $I$ is the center of the circle through $X, Y, Z$ and the sides of the triangle are tangents at $X, Y, Z$. Finally, triangles $AZI$ and $AYI$ are now congruent and so $AI$ is the bisector at $A$.

We note in passing that we have fulfilled an earlier promise to construct a circle that is touched by three given lines. Notice also in the above diagram that we get three cyclic quadrilaterals (why?) and hence that triangle $XYZ$ has angles $(\angle B + \angle C)/2$, $(\angle C + \angle A)/2$, $(\angle A + \angle B)/2$. Notice also that $\triangle ABC = sr$, where $s$ is the usual semiperimeter and $r$ is the inradius. [Recall that we use $R$ for the circumradius.] Combining with an earlier area formula, we get $4srR = abc$, or $rR = (1/2)abc/(a + b + c)$.

Essentially the same argument shows that the internal bisector at $A$ meets the external bisectors at $B$ and $C$ at the center of an excircle which touches $BC$ and $AB, AC$ extended. We leave you to draw the large picture. We get three excircles with centers $I_1, I_2, I_3$ and radii $r_1, r_2, r_3$. We have $r_1 = \Delta/(s-a)$, with analogous formulas for $r_2, r_3$. This formula follows from the equation: $\Delta ABC = \Delta ABI + \Delta ACI - \Delta BCI$. Now we get the delicious formula:

$$rr_1r_2r_3 = \Delta^4/[s(s-a)(s-b)(s-c)] = \Delta^2.$$  

Moreover,

$$(1/r_1) + (1/r_2) + (1/r_3) = [(s-a) + (s-b) + (s-c)]/\Delta = s/\Delta = 1/r.$$
We leave you to show that \( r_1 + r_2 + r_3 = 4R + r \), and also that the sum of the distances from the circumcenter to the sides of the triangle is \( R + r \). From each vertex of the triangle there are two tangents to the opposite excircle; the distance to the point of tangency is \( s \). If side \( AB \) touches the incircle at \( T \) and the excircle at \( T_1 \), then \( TT_1 = a \).

After this flood of amazing formulas we turn quietly to the altitudes.

*The altitudes of a triangle are concurrent at the orthocenter.*

Proof 1. Draw altitudes \( BM, CN \) to intersect at \( H \); let \( AH \) meet \( BC \) at \( L \). Quadrilaterals \( ANHM \) and \( BNMC \) are cyclic. It follows that

\[
\angle NAH = \angle NMH = \angle NCL.
\]

Hence quadrilateral \( ANLC \) is cyclic and so \( \angle CLA = \angle CNA \), a right angle.

Proof 2. Through each vertex of the triangle draw a line parallel to the opposite side to form enveloping triangle \( \triangle PQR \), as in the diagram. Show that \( PSfrags \)

\[
A, B, C \text{ are the mid-points of the sides of triangle } \triangle PQR. \text{ Thus the altitudes}
\]

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of triangle $ABC$ are the perpendicular bisectors of the sides of triangle $PQR$ and so the altitudes are concurrent at the circumcenter of triangle $PQR$.

Proof 3. Use vector notation as in the proof of the concurrence of the medians; thus $O$ is the circumcenter of triangle $ABC$ and the directed line segments $OA, OB, OC$ represent the vectors $\vec{u}, \vec{v}, \vec{w}$ respectively. Let $H$ be the point such that the directed line segment $OH$ represents the vector $\vec{u} + \vec{v} + \vec{w}$. Then $AH$ represents $\vec{v} + \vec{w}$ and $BC$ represents $-\vec{v} + \vec{w}$. Since $\vec{v}$ and $\vec{w}$ each have length $R$, it follows that $AH \perp BC$. Clearly $H$ lies on each altitude. There is a wonderful corollary from this proof. Since $OG$ represents the vector $(1/3)(\vec{u} + \vec{v} + \vec{w})$, it follows that $O, G, H$ are collinear and $OG = (1/3)OH$. This fact is due to Euler and we give a pure proof below.

Notice that each of triangles $AMN, BNL, CLM$ is similar to triangle $ABC$. Also the circles with diameters $AH, BH, CH$ have a “triple” point at $H$ and double points at $L, M, N$. Triangle $LMN$ is called the orthic triangle for the given triangle $ABC$. Use cyclic quadrilaterals to show that the altitudes of triangle $ABC$ are the internal bisectors of the orthic triangle. Thus, in relation to the earlier diagram we have the remarkable sequence; the incenter of $LMN$ is the orthocenter of $ABC$ is the circumcenter of $PQR$.

The centroid and the incenter both lie within the given triangle. This is not the case for the orthocenter (or the circumcenter). For which triangles does the orthocenter (circumcenter) lie outside the triangle?

There are other links with our previous notions. In the diagram below the altitudes meet the circumcircle again at $A', B', C'$. Then $L$ is the mid-
point of $HA'$; similarly for $M, N$. For this, note that $\angle HA'C = \angle ABC = \angle A'HC$. Since $CL$ is perpendicular to $HA'$, we are done. Note also that $A$ is the mid-point of arc $B'C'$ (and similarly) since $\angle C'CA = \angle B'BA$. Since triangles $HBC$ and $A'BC$ are congruent, it follows that the circumradius of triangle $HBC$ is $R$ (and similarly for $HCA$ and $HAB$). The mid-point result shows that $MN$ is parallel to $B'C'$. Since $A$ is the mid-point of arc $B'C'$, we have $OA$ perpendicular to $B'C'$ and so to $MN$ (where $O$ is of course the circumcenter of triangle $ABC$). Thus the radii $OA, OB, OC$ are perpendicular to the corresponding sides of the orthic triangle.

Let $h_1, h_2, h_3$ be the three altitudes. Since $2\Delta = 2sr = ah_1 = bh_2 = ch_3$, it follows that

\[
(1/r) = (1/h_1) + (1/h_2) + (1/h_3).
\]

Compare with earlier formulas!

Let’s revisit two of our fundamental diagrams again. Given triangle $ABC$, draw the circles on diameters $BC, CA, AB$. Then they intersect in pairs at the feet of the altitudes (giving another way to construct these points!). Now prove the remarkable fact that the sum of the areas of the three external curved triangles minus the area of the internal curved triangle is twice the area of triangle $ABC$!

Draw a circle of radius $R$ and choose a triangle $ABC$ with vertices on this circle. Draw the other circle of radius $R$ through $B, C$; similarly for two other circles. Then these three circles intersect at the orthocenter of the triangle (recall earlier result). This is highly relevant to the “silver dollar” problem 177. Note also that $A$ is the orthocenter of triangle $HBC$; similarly for $B, C$. So, if we start with three non-collinear points and “keep drawing new orthocenters” we never get past four points!
For any triangle $ABC$, $O, G, H$ are collinear in the Euler Line of the triangle. In the diagram, draw diameter $COL$. Since $\angle LAC$ is a right angle, $LA \parallel BH$, and similarly $BL \parallel HA$. Hence $AH = BL = 2OA'$. But $AG = 2GA'$ and $\angle HAG = \angle GA'O$. So triangles $HAG$ and $OA'G$ are similar. It follows that $HG = 2GO$ and $H, G, O$ are collinear.

We have had three cases of concurrences of lines which emanate from the vertices of a triangle - medians, altitudes, internal bisectors. They are all special cases of the following theorem.

**CEVA’S THEOREM** Let $P, Q, R$ be points on $BC, CA, AB$ of triangle $ABC$. Then $(BP : PC)(CQ : QA)(AR : RB) = 1$ if and only if $AP, BQ, CR$ are concurrent.

Suppose the lines are concurrent at $S$.

Proof 1. Let $CX, BY$ be parallel to $PS$, as in the diagram below. By similar triangles, $BP : PC = YS : SC = BY : CX$, and in the same way,

Proof 2. We have $BP : PC = \Delta BAP : \Delta CAP = \Delta BSP : \Delta CSP = \Delta ASB : \Delta ASC$, with a similar formula for the other two ratios. The result follows.

Conversely, suppose the product of the ratios is 1. Let $BQ, CR$ meet at $S$, and let $AS$ meet $BC$ at $P'$. It follows from the first part and the given condition that $BP : PC = BP' : P'C$. Hence $P = P'$ and the proof is complete.

Use Ceva to reprove our three known cases; the median case is trivial, use similar triangles for the altitudes, and use the Sine Rule for the internal bisectors. As a new application, let the incircle touch $BC, CA, AB$ at $X, Y, Z$ respectively. Then $AX, BY, CZ$ are concurrent (at the Gergonne point); use the equality of tangents from a point to a circle! Is there another concurrence obtained from the points on $BC, CA, AB$ that are touched by the ex-circles?

The “point-line” duality suggests that there might be a corresponding theorem with three points collinear. The three points cannot all lie on the actual sides of the triangle — either one point or all three points must be on the extended sides. For this reason, it is here convenient to consider directed ratios.

**MENELAUS’ THEOREM** Let $L, M, N$ lie on the lines determined by sides $BC, CA, AB$ respectively of triangle $ABC$. Then $L, M, N$ are collinear if and only if $(BL : LC)(CM : MA)(AN : NC) = -1$.

Proof. Suppose $L, M, N$ are collinear. Drop perpendiculars from $A, B, C$ to the line $LMN$ as in the diagram. Apply similar triangles to get the product of the ratios as $-1$. For the converse use the strategy analogous to that used in the converse of Ceva’s Theorem.
The following remarkable theorem is a high point of the geometry of the triangle. Dame Mary Cartwright recalled that, when she went for interview for admission to undergraduate mathematics at the University of Cambridge, she had to give two different proofs of the theorem!

**THE NINE POINT CIRCLE** In a triangle, the mid-points of the sides, the feet of the altitudes, and the points mid way between the orthocenter and the vertices lie on a circle with center at the mid-point of $OH$ and radius $R/2$.

Proof 1. The circle on diameter $AB$ contains $D$ and so $C'D = C'B = A'B'$ (see next page). Since $B'C' \parallel A'D$, quadrilateral $B'C'DA'$ is thus cyclic. So $D$ lies on the circle through $A', B', C'$, and similarly for $E, F$. Since $C'P \parallel BE$, $\angle A'C'P$ is a right angle, and so quadrilateral $PC'DA'$ is cyclic. So $P$ lies on the circle through $A', C', D$ (i.e. the circle through $A', B', C'$),
and similarly for $Q, R$. Let $N$ be the mid-point of $OH$. The perpendicular bisector of $DA'$ passes through $N$, similarly for $B'E, C'F$. Thus $N$ is the center of the nine point circle. We proved earlier that $AH = 2OA'$. So $OA' = AP$, and hence $OAPA'$ is a parallelogram. Therefore, $OA = A'P$; but $A'P$ is a diameter of the nine point circle.

INTERLUDE ON COMPLEX NUMBERS

Recall that we write a complex number as $z = x + iy$ where $x, y$ are real numbers and $i^2 = -1$. Here, we write the complex conjugate of $z$ as $z^* = x - iy$. Thus,

$$zz^* = x^2 + y^2 = |z|^2 = r^2.$$  

We also have the polar form $z = re^{i\theta}$, where $x = r \cos(\theta)$ and $y = r \sin(\theta)$.

Consider the line in the plane given by $ax + by = c$. We can express $x, y$ in terms of $z, z^*$ and then the equation can be written as

$$\alpha^* z + \alpha z^* = 2c$$

where $\alpha = a + ib$. Given complex numbers $\alpha, \beta, \gamma$, not every equation of the form $\alpha z + \beta z^* = \gamma$ represents a line. In fact when we equate real and imaginary parts in this last equation we see that it is equivalent to two real simultaneous linear equations; thus we get a line for the locus if and only if the two real equations determine the same line. But the other possibilities are that we have parallel lines or else two lines that intersect in just one point. So we see that the equation $\alpha z + \beta z^* = \gamma$ represents a line if and only if the equation is satisfied by two distinct values of $z$.

Given distinct points $z_1$ and $z_2$, the line through them is given in determinantal form by

$$\begin{vmatrix} z & z^* & 1 \\ z_1 & z_1^* & 1 \\ z_2 & z_2^* & 1 \end{vmatrix} = 0.$$  

The slope of the line $z + \alpha z^* = \gamma$ depends only on $\alpha$ (indeed it is parallel to the line $z + \alpha z^* = 0$ through the origin). The lines

$$z + \alpha z^* = \gamma, \quad z + \beta z^* = \delta$$

are perpendicular if and only if $\alpha + \beta = 0$. To see this, recall that multiplication by $i$ rotates the plane through 90 degrees. Replace $z$ by $iz$ in the first equation and the result is immediate.
Many other aspects of geometry can be rewritten in terms of complex numbers, but the above is sufficient for our immediate purposes. Other ideas are given in the problem list.

Proof 2. Without loss we may suppose (why?) that $R = 1$. With the origin at $O(!)$, we can describe vertices $T_1, T_2, T_3$ by $t_1, t_2, t_3$ of absolute value 1. Write $s_1 = t_1 + t_2 + t_3, s_2 = t_2 t_3 + t_3 t_1 + t_1 t_2, s_3 = t_1 t_2 t_3$. Verify that the mid-point of the side opposite $T_i$ is given by $m_i = (s_1 - t_i)/2$, and that the side opposite $T_i$ is given by $z + (s_3/t_i)z^* = s_1 - t_i$. Now show that the altitude through vertex $T_i$ has equation $z - (s_3/t_i)z^* = s_1 - t_i$. It follows that the orthocenter $H$ is given by $s_1$. (We already knew this from our earlier vector discussion of the orthocenter; of course the centroid is $s_1/3$.) Show next that the foot of the altitude through $T_i$ is given by $h_i = (s_1 - s_3/t_i^2)/2$. The mid-point of $HT_i$ is of course given by $f_i = (s_1 + t_i)/2$. The nine points $m_i, h_i, f_i$ all clearly satisfy the equation $|z - s_i/2| = 1/2$.

Now we show some other applications of the use of complex numbers. First we reprove Ptolemy’s Theorem. In quadrilateral $ABCD$ let the directed sides $AB, BC, CD, DA$ represent complex numbers $\alpha, \beta, \gamma, \delta$. Notice that $\alpha + \beta + \gamma + \delta = 0$. We have

$$AC.BD = |\alpha + \beta||\beta + \gamma| = |\alpha \gamma + \beta (\alpha + \beta + \gamma)| = |\alpha \gamma - \beta \delta| \leq AB.CD + BC.AD.$$  

We leave you to produce a complex number argument to show that equality holds if and only if $ABCD$ is cyclic. Many other geometric inequalities can be obtained by using algebraic identities. Here is one illustration, others are in the exercises. The Vandermonde $3 \times 3$ determinant leads to the identity

$$-(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta) = \alpha^2(\beta - \gamma) + \beta^2(\gamma - \delta) + \gamma^2(\alpha - \beta).$$

It follows that

$$|\beta - \gamma||\gamma - \alpha||\alpha - \beta| \leq |\alpha|^2|\beta - \gamma| + |\beta|^2|\gamma - \alpha| + |\gamma|^2|\alpha - \beta|.$$  

Now let $A, B, C$ be any three points in the plane that represent $\alpha, \beta, \gamma$ with respect to some origin $O$. Then we get

$$BC.CA.AB \leq OA^2.BC + OB^2.CA + OC^2.AB.$$  

When $ABC$ is an equilateral triangle and $O$ is any point in the plane, we thus get $OA^2 + OB^2 + OC^2 \geq d^2$ where $d$ is the side length of the equilateral triangle. When $ABC$ is any triangle and $O$ is the circumcenter we get

$$abc \leq R^2(a + b + c).$$
Using two known formulas for the area of the triangle we deduce that $R \geq 2r$.

We end this section with an exercise that is considered to be a “classical chestnut”. If $ABC$ is a triangle with $AB = AC$, then the angle bisectors at $B$ and $C$, say $BP$ and $CQ$, are obviously equal in length by symmetry. How about the converse? If the angle bisectors at $B$ and $C$ are equal, do we have $AB = AC$? The answer is yes, but pure proofs are notoriously tricky. Here is a sketch of a proof by trigonometry. Call the angles of the triangle $\alpha, \beta, \gamma$. We warm up on an easier converse; if $BQ = CP$ then $\beta = \gamma$. For this one, apply the Sine Rule to triangles $PBC$ and $QBC$ to get

$$\sin(\alpha + \gamma/2) \sin(\beta/2) = \sin(\alpha + \beta/2) \sin(\gamma/2).$$

Write each side as a difference of cosines to deduce that $\cos \lambda = \cos \mu$ where we have $\lambda = \alpha + (\beta - \gamma)/2$, $\mu = \alpha - (\beta - \gamma)/2$, and thence $\beta = \gamma$, as required.

For the actual converse, with $BP = CQ$, we again apply the Sine Rule to triangles $PBC$ and $QBC$; but now we get the equation

$$\sin \beta \sin(\gamma + \beta/2) = \sin \gamma \sin(\beta + \gamma/2).$$

We suppose that $\beta \neq \gamma$ and eventually produce a contradiction. Without loss of generality, we take $\beta > \gamma$. We symmetrize by writing $\theta = (\beta + \gamma)/2, \delta = (\beta - \gamma)/2$ and note that $\pi/2 > \delta > 0$. Our equation now becomes

$$\sin(\theta + \delta) \sin(3(\theta - \delta)/2) = \sin(\theta - \delta) \sin(3(\theta + \delta)/2).$$

Expand, using $\sin(A \pm B)$, and cancel like terms to get

$$\sin \theta \cos(3\theta/2) \cos \delta \sin(\delta/2) = \sin(3\theta/2) \cos \theta \sin \delta \cos(\delta/2).$$

Gather $\theta$’s and $\delta$’s on separate sides to get

$$(\sin \theta \cos(3\theta/2)) / (\sin(3\theta/2) \cos \theta) = 2 \cos^2(\delta/2) / \cos \delta = 1 + 1 / \cos \delta$$

and hence

$$1 / \cos \delta = (\sin \theta \cos(3\theta/2) - \cos \theta \sin(3\theta/2)) / \sin(3\theta/2) \cos \theta$$

$$= -\sin(\theta/2) / (\sin(3\theta/2) \cos \theta).$$

This gives

$$-\cos \delta = \cos \theta (3 - 4 \sin^2(\theta/2)) = \cos \theta (3 - 2(1 - \cos \theta)) = \cos \theta (1 + 2 \cos \theta).$$

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But $\theta = (\beta + \gamma)/2 < \pi/2$, so that $\cos \theta > 0$. Since also $\cos \delta > 0$, the desired contradiction follows from the last equation. But this is hardly an easy proof! In fact, it is included as a joke against the teacher - a much easier argument is outlined in Problem 215.
Pole-polar duality is usually defined by pure geometry. We here begin with the coordinate geometry viewpoint since we get the key results very simply and quickly. We are given a circle and two points $P, Q$. Take the origin at the center of the circle; so that the circle is given by $x^2 + y^2 = r^2$ and we may take $P = (x_1, y_1), Q = (x_2, y_2)$. We say that $P, Q$ are polar with respect to the circle if $x_1x_2 + y_1y_2 = r^2$. We define the polar line of $P$ by $x_1x + y_1y = r^2$. It is immediate that $Q$ lies on the polar line of $P$ if and only if $P$ lies on the polar line of $Q$. The discussion would appear to depend on where we place the $x$-axis, but this is not so. In complex notation our condition is $Re(z_1z_2^*) = r^2$. For a rotation through $\theta$, we have $z_1z_2^* = (e^{i\theta}z_1)(e^{i\theta}z_2)^*$ and so the algebraic condition for polarity is independent of the choice of the $x$-axis. [Notice that there is no polar line for the center of the circle - in projective geometry the polar of the center would be the line at infinity.]

If $P$ lies on the circle, the polar line is just the tangent at $P$. Let $P$ be outside the circle and let the tangents to the circle from $P$ meet the circle at $Q, R$. $P$ lies on the polar line of $Q$ and so $Q$ lies on the polar line of $P$. Similarly $R$ lies on the polar line of $P$. So the polar line of $P$ is the line through $Q, R$. Now let $P'$ be inside the circle (without loss on the $x$-axis as shown). Take the chord $QR$ perpendicular to $OP'$ and let the tangents at $Q, R$ meet at $P'$. Then $P$ lies on the polar line of $P'$. Since the polar line is of the form $x = \text{const.}$, it is the line through $P$ perpendicular to $OP$. By similar triangles we have $OP' : OQ = OQ : OP$ and hence $OP.OP' = r^2$. Deduce that the diameter $AB$ is divided internally and externally in the same ratio at $P'$ and $P$. We then say that $P, P'$ are harmonic conjugates with respect to $A, B$. If we use directed ratios, this is equivalent to asking that the cross-ratio $(AP' : P'B)/(AP : PB) = -1$.

The pole-polar duality sends points to lines and lines to points; so a triangle is transformed to a triangle (vertices to sides). Can we have a self-
polar triangle (in which the polar of each vertex is the side opposite that vertex)? Let $A$ be inside the circle as shown and choose any $B$ on the polar line of $A$. The polar line of $B$ passes through $A$ and meets the polar line of $A$ at $C$. Then triangle $ABC$ is self-polar. Since $OB$ is perpendicular to $AC$, $\angle BAC$ is obtuse. Show conversely that given any obtuse triangle $ABC$ we can find a circle for which $ABC$ is self-polar and that the center of the circle is at the orthocenter of triangle $ABC$. Show that the orthocenter has to lie outside any self-polar triangle so that a self-polar triangle has to be obtuse. For a nice non-self-dual picture, take a triangle $ABC$ inscribed in its circumcircle, center $O$. The tangent lines to the circle at $A, B, C$ form a triangle $PQR$ which is the dual of triangle $ABC$. Notice that $O$ is the incenter of triangle $PQR$. This adds yet another triangle to the interesting sequence of triangles discussed on page 354.

With a point-line duality we should expect to pass between theorems on concurrence and theorems on collinearity. Let $ABC$ be a triangle inscribed in a circle. Draw chords through $A, B, C$ and let the tangents at the ends of these chords meet at $L, M, N$. Show that the three chords are concurrent at $P$ if and only if $L, M, N$ are collinear (on the polar line of $P$).

An even more important duality comes from a point-point relationship which we already saw above. Given a circle center $O$, radius $r$, and $P, P'$ on the ray from $O$, we say that $P, P'$ are inverse points with respect to the circle if $OP.OP' = r^2$; equivalently, $P, P'$ are harmonic conjugates with respect to the ends of the diameter formed by the ray. The situation has a simple description via complex multiplication; if $P = z$ then $P' = r/z^*$. We call the map $P$ to $P'$ inversion (with respect to the given circle). Notice that if we invert $P'$ we get back to $P$. Inversion has the remarkable property that it maps circles to circles (here we regard lines as degenerate circles with center at infinity). Suppose that $|z - \alpha| = \rho$ or equivalently $zz^* - \alpha z^* - \alpha^* z + \alpha \alpha^* = \rho^2$. Let
\( w = r^2/z^* \) and check that \( w \) satisfies the equation

\[
(|\alpha|^2 - \rho^2)ww^* - r^2\alpha w^* - r^2\alpha^* w + r^4 = 0
\]

a circle. The circle is degenerate if \(|\alpha| = \rho\), i.e. the center of the circle of inversion lies on the first circle. You should check that if \( z \) travels in the clockwise direction then \( w \) travels in the counter-clockwise direction. Of course a line inverts to a circle since inversion is its own inverse! It is also trivial that the circle of inversion inverts to itself.

We can easily see geometrically how to get the inverted circle. Let the circle of inversion have center \( O \) and the first circle center \( C \). Join \( OC \) to get the diameter \( AB \) of the first circle, and then get the inverse points \( B', A' \) as shown. The inverted circle has \( B'A' \) as diameter. Pleasant geometrical properties abound. If \( P, P' \) and \( Q, Q' \) are inverse points then the equation \( OP.OP' = OQ.OQ' \) makes the quadrilateral \( PP'QQ' \) cyclic. Any circle through \( P, P' \) cuts the circle of inversion at right angles (this is harder work and we leave it to you).

Perhaps the most important property of inversion is that it preserves angles between curves (though reversing the orientation), i.e. if two lines or circles intersect at angle \( \alpha \) then so do the image curves. This is essentially obvious from the association of \( z \) with \( r/z^* \). As a simple application, note that tangency is preserved by inversion; e.g. if two circles are tangent at \( P \) then the inverted circles are tangent at \( P' \).

A triangle inverts into three circles with a common point (the center of the circle of inversion). Note that we now have a one line proof of Problem 45. Any theorem that involves only lines, circles, intersections and angles can be "inverted" into a dual theorem. For example, take the concurrence of
the altitudes of a triangle $ABC$ at $H$. What theorem do we get if we invert with respect to the circumcircle of $ABC$? What theorem do we get if we invert with respect to the circle on diameter $AH$? What do we get from the concurrence of the internal bisectors if we invert with respect to the incircle? What about more elementary properties of the circle — such as the equality of angles on the same chord?

Here is yet another variation on the concurrence of altitudes theme. Three circles meet at $O$ and intersect in pairs at $A, B, C$. The lines $OB, OC$ pass through the center of the other circle. Prove that $OA$ also passes through the center of the other circle. [Invert with respect to a circle center $O$ and use the concurrence of the altitudes in the inverted figure.]

It should be clear that the choice of the circle of inversion is critical. With a clever choice of the circle of inversion the inverted figure may become particularly simple or tractable. It is important to know how much freedom we have to adjust circles under an inversion. Can we invert three circles so that the new circles all have the same radius? Can we invert three circles so that the new centers are collinear? We leave the deeper study of inversion to the next course in geometry.
8 A FEW GEMS FROM HERE AND THERE

In this final section we present a pot-pourri of results old and new. We begin with a new result (post World War II!) which the author called the most elementary theorem in Euclidean geometry (Australian sense of humor).

URQUHART’S THEOREM Let $A, B, C$ and $A, D, E$ be collinear and let $CD, BE$ meet at $F$. Then we have $AD + DF = AB + BF$ if and only if $AE + EF = AC + CF$.

Proof. Those who like coordinate geometry may care to do this by algebra - yes, it can be done but take a large supply of paper and determination. In one sense it is a theorem about ellipses; it says that $B, D$ lie on an ellipse with foci $A, F$ if and only if $C, E$ lie on an ellipse with foci at $A, F$. A proof is available this way using the modern theory of dual billiards. There are several pure geometry proofs available. The following is our favorite (it took more than half an hour to discover!). Without loss we suppose that $AD + DF = AB + BF$. Let $B', D'$ be as in the diagram so that $BF = BB', DF = DD'$. Then we have $AB' = AD'$. Similarly we take $C', E'$ in the diagram so that $CF = CC', EF = EE'$. Let $O$ be the circumcenter of triangle $B'FD'$. Then $O$ lies on the angle bisectors at $A, B, D$, as shown. By exterior angles, $\angle ACD = 2\beta - 2\alpha$ and so $\angle B'C'F = \beta - \alpha$ (triangle $FCC'$ is isosceles). But $\angle B'D'F = (\pi/2 - \alpha) - (\pi/2 - \beta) = \beta - \alpha$. Therefore $C'$ lies on the circle through $B', F, D'$ and similarly does $E'$. By exterior angles we get

$$\angle B'FC' = (\pi/2 - \gamma) - (\beta - \alpha) = (\pi/2 - \beta) - (\gamma - \alpha) = \angle D'FE'.$$
Since chords $B'C',D'E'$ subtend equal angles at the circle we get $B'C' = D'E'$ and so $AC' = AE'$, $AC + CF = AE + EF$, as required. As an addendum for the diagram itself, note that $\angle FOB' = 2\angle FC'C = 2\beta - 2\alpha = \angle FCB$. Thus $FB'CO$ is a cyclic quadrilateral and similarly for $FD'EO$.

We now go back to the 18th century for the Simson line (apparently discovered by Wallace after the death of Simson!). Let $P$ be any point on the circumcircle of triangle $ABC$. Let $L,M,N$ be the feet of the perpendiculars from $P$ to the sides of the triangle. Then $L,M,N$ are collinear in the Simson line (or pedal line) for $P$. By cyclic quadrilaterals in the diagram we get

$$\angle PNM = \angle PAM = \angle PBC$$

and the latter is supplementary to $\angle PNL$. Hence $LMN$ is a straight line. As usual let $H$ be the orthocenter of triangle $ABC$. Then $PH$ is bisected by the Simson line. To see this, extend altitude $CE$ to meet the circle in $T$ and extend $PN$ to meet the circle in $K$. Draw $HU$ parallel to $CK$ with $U$ on $PK$. $PK,TC$ are both perpendicular to $AB$ and so are parallel. Thus $CHUK$ is a parallelogram and so $HU = CK = PT$.

From an argument in Section 6, $E$ is the mid-point of $TH$ and so $N$ is the mid-point of $PU$. Since $\angle PKC$ is supplementary to $\angle PBC = \angle PNM$, we have $LN \parallel CK \parallel HU$. Hence $LN$ bisects $PH$, at $X$, say. The center of the nine-point circle $N$ is the mid-point of $OH$ and the nine-point circle has radius $R/2$. Hence $X$ lies on the nine-point circle; as $P$ traverses the circumcircle, $X$ traverses the nine-point circle.

Let $P'$ be another point on the circumcircle. Then the angle between the pedal lines of $P$ and $P'$ is half the arc $PP'$. To see this, let the perpendicular
from $P'$ to $AB$ meet the circle again at $K'$. The pedal line of $P$ is parallel to $CK$ and that of $P'$ is parallel to $CK'$. So the angle between the pedal lines is $\angle KCK'$ which is half arc $KK'$. But arc $KK'$ is equal to arc $PP'$. In particular, if $P, P'$ are diametrically opposite then their pedal lines are perpendicular — moreover these pedal lines intersect on the nine-point circle! For if $X, X'$ are the mid-points of $PH, P'H$ then we find that $XX'$ is a diameter of the nine-point circle. But the pedal lines pass through $X, X'$ and are perpendicular and so the intersection lies on the circle with diameter $XX'$, as claimed. It is a remarkable fact that as $P$ traverses the circle the pedal lines envelope a three-pointed cusp (in fact a hypocycloid).

We leave you to verify the converse of the original assertion; namely if $P$ is such that the feet of the perpendiculars from $P$ to $BC, CA, AB$ are collinear then $P$ lies on the circumcircle of triangle $ABC$. Another nice exercise is to show that if $P, Q, R$ are points on the circumcircle, then the triangle formed by their pedal lines is similar to triangle $PQR$.

We can use some of the above ideas to solve some problems associated with the complete quadrilateral formed by four lines as shown. (Actually for the complete quadrilateral we should join up the inner diagonals as well.) The circumcircles of the four triangles have a common point. Let the circ-

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F
\end{array}
\]

\[
\begin{array}{c}
P \\
Q \\
R \\
S
\end{array}
\]

cumcircles of $CDF, CBE$ meet again in $M$. Let $P, Q, R, S$ be the feet of the perpendiculars from $M$ to the given four lines. By Simson (twice), $P, Q, R, S$ are collinear. By the converse of Simson (twice), $M$ lies on the circumcircle of $ABF$ and $ADE$. We also have that the orthocenters of these four triangles are collinear. For the lines joining $M$ to the four orthocenters are bisected by the common Simson line $PQRS$, and hence the four orthocenters are collinear.
Consider now the complete quadrilateral drawn overleaf and let \( FP \) meet \( AB \) at \( E' \). Then \( E, E' \) are harmonic conjugates with respect to \( A, B \). It follows (prove it!) that any transversal of \( FA, FE', FB, FE \) gives a harmonic range of points. To prove the first assertion, apply Ceva’s Theorem to triangle

\[
\begin{align*}
\text{PSfrag replacements} \\
A & \quad B & \quad C & \quad D & \quad E & \quad F \\
\text{D} & \quad \text{P} & \quad \text{C} & \quad \text{e} & \quad \text{B} & \quad \text{E} \\
\end{align*}
\]

\( ABB \) with point of concurrence \( P \), apply Menelaus’ Theorem triangle \( ABF \) for the line \( ECD \), and divide to get \((AE' : E'B)/(AE : EB) = -1\), as required. Can you see how this leads to a straightedge construction for the harmonic conjugate of \( E \) with respect to \( A, B \)?

We move forward to the late nineteenth century and the American mathematician Frank Morley. He discovered that the intersections of adjacent pairs of angle trisectors in a triangle are the vertices of an equilateral triangle. The result spread orally though an elementary proof did not appear until 1914. We shall give one of the pure proofs and then sketch the standard trigonometric proof.

**MORLEY’S THEOREM** Adjacent angle trisectors of any triangle \( ABC \) meet in the vertices of an equilateral triangle.

Proof. This pure proof is not elegant but it does make a direct attack at calculating the key angles in the following complicated diagram. \( PQR \) is the Morley triangle and so \( R \) is the incenter of triangle \( ABL \). The incircle meets \( BL, AL \) at \( S, T \), and \( RS, RT \) meet \( BC, CA \) at \( M, K \). Let the tangent from \( K \) meet the incircle at \( V \) and \( BL \) at \( F \).
We are going to show that $F = P$. Note that $RT = TK = RS = SM$. So $RM = RK$ and $RV = RK/2$, and hence $\angle VRT = \pi/3, \angle RKF = \pi/6$. Let $\alpha, \beta, \gamma$ be one third of the angles $A, B, C$ respectively. We have $\angle SRT = \pi - \angle SLT = 2\alpha + 2\beta = 2\pi/3 - 2\gamma$, and hence $\angle FMS = \angle FRS = \angle VRS/2 = (\angle SRT - \angle VRT)/2 = \pi/6 - \gamma$. Next, $\angle RMK = \angle RKM = (\pi - \angle SRT)/2 = \pi/6 + \gamma$. This gives $\angle FMK = 2\gamma, \angle FKM = \gamma$ and so $\angle MFK$ is supplementary to $\angle MCK$, so that $MCKF$ is a cyclic quadrilateral. Then $\angle FCM = \angle FKM = \gamma = \angle PCM$, and so finally we have shown that $F = P$. Similarly the tangent from $M$ meets the incircle at $U$ and then passes through $Q$. Since triangle $RMK$ is isosceles and $MQ, KP$ are symmetrical it follows that $\angle SRP = \angle PRV = \angle UQ = \angle QRT = \pi/6 - \gamma$. Since $\angle VRT = \pi/3$, it is immediate that $\angle PRQ = \pi/3$. Similarly we get $\angle PQR = \pi/6 = \angle RPQ$.

For the trigonometric argument, apply the Sine Rule to each of the triangles $ARQ, ARB, AQC$ to deduce, with $x = \angle ARQ, y = \angle AQR$, that

$$\sin x : \sin y = [\sin \gamma \sin 3\beta \sin(\alpha + \beta)] : [\sin(\alpha + \gamma) \sin 3\gamma \sin \beta]$$

and hence

$$\sin x : \sin y = [\sin \gamma \sin(\pi/3 - \gamma) \sin 3\beta] : [\sin \beta \sin(\pi/3 - \beta) \sin 3\gamma].$$
The identity: \( \sin \theta = 4 \sin(\theta/3) \sin((\pi + \theta)/3) \sin((\pi - \theta)/3) \) now gives
\[
\sin x : \sin y = \sin(\pi/3 + \beta) : \sin(\pi/3 + \gamma).
\]
Since \( 0 < x + y = \pi/3 + \beta + \pi/3 + \gamma < \pi \), a little trigonometric lemma
now forces \( x = \pi/3 + \beta, y = \pi/3 + \gamma \). Similarly we calculate each of the
angles \( \angle BRP, \angle BPR, \angle CPQ, \angle CQP \) and then a routine angle computation
completes the proof.

Since both of the above proofs are fairly tough we shall give a third proof,
which is Coxeter’s polished version of an idea first given by Roger Penrose
in 1953. [Coxeter and Penrose are both outstanding modern geometers.] The other reason for giving this proof is that it is an example of working
backwards; we start with an equilateral triangle and construct a triangle for
which this is the Morley triangle and note that the method thereby constructs
all possible triangles. Given equilateral triangle \( PQR \), we construct exter-
nal isosceles triangles as indicated in the diagram overleaf, with base angles
\( \alpha, \beta, \gamma \) such that \( \alpha + \beta + \gamma = 2\pi/3 \) and each angle is less than \( \pi/3 \). Extend
the sides of these isosceles triangles to form triangle \( ABC \) as shown. Since
\( \alpha + \beta + \gamma + \pi/3 = \pi \), we can calculate the other marked angles in the diagram.

Also \( \angle RAQ = \pi/3 - \alpha, \angle PBR = \pi/3 - \beta, \) and \( \angle PCQ = \pi/3 - \gamma \). Note that
$P'P$ bisects $\angle BP'C$. Review our discussion of the incenter to see that $P$ will be the incenter of triangle $BP'C$ provided we have $\angle BPC = \pi/2 + \angle B'PC/2$. This last equation is easily verified. So $P, Q, R$ are the incenters of the obvious triangles and hence the angles at $A, B, C$ are all trisected. Thus $PQR$ is the Morley triangle of $ABC$. The angles at $A, B, C$ are $\pi - 3\alpha, \pi - 3\beta, \pi - 3\gamma$ and we can arrange any values for these (with sum $\pi$, of course) by choosing $\alpha, \beta, \gamma$ suitably. The proof is complete.

All good things come to an end, and so we are going to close with some preliminary discussion of some recent pizza problems (posed in initial forms by Konhauser and Klamkin). In a restaurant, a pizza is usually sliced into eight equal pieces by cutting lines of angle $\pi/4$ at the center of the pizza.

If we cut at equal angles through an off-central point, the slices are clearly not all equal, but is the total shaded area the same as the unshaded area? What if we mark equally spaced points around the rim and join these to the off-central point? In both cases the answer is yes. In the second case we can slice off lines at the crust to change to an equivalent problem for the regular octagon; this is now an easy problem which we leave to you. [Is the answer still yes for $2n$ equally spaced points?] The first style of pizza problem is much harder (especially for $2n$ equal angles), and we shall forbear giving the calculus-geometry proof that Isaac Newton would have produced as relaxation late in the evening! Consider the case in which the cut point is a point on the boundary of the pizza (a careless cut indeed!). Draw your own picture and then flip over pieces of the pizza to convert to a zigzag problem. Does the zigzag bisect the area of the circle? You may prove this by using coordinates or by drawing some helpful additional lines! We note that some
general versions of the pizza problem remain unresolved.

We apologize for doing no solid geometry in the text, even though there
are several nice problems available in the Problem List. Equally we have done
nothing in the text on fascinating tiling problems (some are in the Problems).
In fact, we are embarrassed at the number of topics omitted - but there is a
Geometry II course after this one!
APPENDIX

Some axiom schemes for Euclidean Geometry

1. EUCLID

Euclid begins with 23 definitions of geometrical terms. These definitions are largely circular in that one idea is given in terms of other undefined ideas. For example, a point is that which has no part, a line is length without breadth, a straight line is a line which lies evenly with the points on itself, and so on. Of course these are not formal mathematical definitions and we do just as well with our simple intuitive pictures. His postulates are at least brief!

I. A straight line can be drawn from any point to any point.
II. A finite straight line can be produced continuously in a straight line.
III. A circle may be described with any center and distance.
IV. All right angles are equal to one another.
V. If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles.

The last one is called the "parallel postulate" even though it says nothing directly about parallel lines.

2. HILBERT

The undefined terms are point, line, plane, lie (point on a line), between, congruence. The axioms come in five groups.

I. Axioms of incidence
   I-1. For every two points $A, B$ there exists a line that contains each of the points $A, B$.
   I-2. For every two points $A, B$ there is no more than one line that contains each of the points $A, B$.
   I-3. There exist at least two points on a line. There exist at least three points which do not lie on a line.
   I-4. For any three points $A, B, C$ that do not lie on the same line, there exists a plane that contains each of the points $A, B, C$. For every plane, there exists a point which it contains.
II. Axioms of order

II-1. If point $B$ is between points $A$ and $C$, then $A, B, C$ are distinct points on the same line and $B$ is between $C$ and $A$.

II-2. For any two distinct points $A$ and $C$ there is at least one point $B$ on the line $AC$ that is between $A$ and $C$.

II-3. If $A, B, C$ are three points on the same line, then no more than one is between the other two.

II-4. Let $A, B, C$ be three points that are not on the same line and let $\ell$ be a line in a plane containing $A, B, C$ that does not meet any of the points $A, B, C$. Then if $\ell$ passes through a point of the segment $AB$ it will also pass through a point of the segment $AC$ or of $BC$.

III. Axioms of congruence

III-1. If $A, B$ are two points on a line $\ell$, and if $A'$ is a point on the same line, or another line $\ell'$, then it is possible to find a point $B'$ on a given side of the line $\ell'$ such that $AB$ and $A'B'$ are congruent.

III-2. If $A'B'$ and $A''B''$ congruent to $AB$, then $A'B', A''B''$ are congruent to each other.

III-3. On line $\ell$ let $AB, BC$ meet only at $B$. On $\ell$ or another line $\ell'$, let $A'B', B'C''$ meet only at $B'$. If $AB, BC$ are congruent to $A'B', B'C'$ respectively, then $AC$ is congruent to $A'C''$.

III-4. If $\angle ABC$ is any angle and if $B'C'$ is a ray, then there is exactly one ray on each side of $B'C'$ such that $\angle A'B'C' = \angle ABC$. Every angle is congruent to itself (independent of the choice of $A, C$ on the rays from $B$).

III-5. $SAS$ implies congruence of triangles (all the angles become equal).

IV. Axiom of parallels (Playfair’s Postulate)

IV-1. Through a given point $A$ not on a given line $\ell$ there passes at most one line which does not meet $\ell$.

V. Axioms of continuity

V-1. Given segments $AB, CD$ there exists a number $n$ such that $n$ copies of $CD$ constructed contiguously from $A$ along the ray $AB$ will pass beyond $B$.

V-2. No new points can be added to a line without violating one of the above list of axioms.
BIRKOFF

The undefined terms for Birkoff are points, lines, distance and angle.

I. Any line \( \ell \) is isometric to \( \mathbb{R} \).

II. Exactly one line \( \ell \) contains two distinct points \( P, Q \).

III. The angles about any point \( O \) are isometric to \( \mathbb{R} \) (mod \( 2\pi \)).

IV. The SAS similarity condition implies that the triangles are similar (all angles are equal and corresponding sides are proportional).

Hilbert’s axioms (1899) have enough built into them that one can show that each line is isometric to \( \mathbb{R} \). Birkoff’s (1932) are gloriously brief but they need lots of work to get all the usual properties of the plane. A School Mathematics Study Group in 1961 produced 22 axioms for Euclidean geometry based on the undefined terms point, line, plane. Several other mathematicians have produced equivalent axiom schemes during this century. Various attempts were made to adapt some of the axiom schemes for use in high schools (often with subtleties ignored or slurred over); some of us believe that the intuitive approach is adequate!
A PROBLEM LIST FOR GEOMETRY

The following list of problems is partially arranged by theme but also has the appearance of random order. This arrangement is intentional. When we encounter a geometry problem in a magazine or journal we are not told which section of a given textbook it refers to - we are just given the problem to solve! [But to a first approximation the arrangement of the first hundred or so problems follows the order in the text.] There are no warning signs for the level of difficulty of the problems: a few problems are very easy and some are very hard.

No pictures are drawn in these problems. Again this is intentional. The problem solver should build up the required picture in stages and then doodle away with it. This is the best way to get engaged in problem solving (even though we lose a few eye-catching pictures for the reader).

PROBLEMS THAT NEED ONLY (?) SCHOOL GEOMETRY

1. Is it possible to draw 5 lines on a plane so that each line intersects exactly 3 other lines? Does the conclusion change if we replace straight lines by arbitrary paths? Is it possible to join 77 points so that each is connected to exactly 15 other points?

2. Given the point \((a, b)\), what is a (pure) geometrical construction that gives the line \(ax + by = 1\)? [Hint: Begin by considering a suitable collection of special cases for the point \((a, b)\), such as \((a, 0)\), \((0, b)\), \((a, 2a)\), etc.]

3. Verify the triangle inequality for distance as defined by the usual Euclidean formula. [Hint: Show first, by translation, that we may suppose without loss that one of the three points is at the origin. You will want to use repeatedly the fact that, for non-negative \(t\), we have \(t \geq 0\) if and only if \(t^2 \geq 0\).]

4. Given a segment \(AB\) and a line \(\ell\), let \(n(\ell)\) be the number of points \(C\) on \(L\) such that triangle \(ABC\) is isosceles. What are the possible values of \(n(\ell)\)?

5. In triangle \(ABC\), \(AB = 10\) and \(AC = 15\). The bisector of angle \(A\) meets \(BC\) at \(D\). Show that \(AD \leq 12\).

6. Five points are given on an infinite grid of unit squares and we mark the mid-points of the ten segments joining pairs of these five points. Prove that at least one of these mid-points is a grid point.

7. We are given a fixed triangle \(ABC\) and we choose any point \(D\) on \(BC\) and then the two points \(X\) such that triangle \(ADX\) is equilateral. Describe
the set of all such points $X$ as $D$ varies along $BC$.

8. Outside triangle $ABC$ we draw isosceles right triangles $ABD$, $ACE$ with hypotenuses $AB$, $AC$. Let $F$ be the mid-point of $BC$. Prove that $DFE$ is an isosceles right triangle.

9. The three vertices of an equilateral triangle lie on three parallel lines, the middle line at distances $b, c$ from the outer lines. If $x$ is the length of a side of the triangle, show that $3x^2/4 = b^2 + bc + c^2$. Generalize to the case of three adjacent vertices of a regular polygon.

10. What is the largest number of different regions into which the plane can be divided by $n$ lines?

CONGRUENCE AND SIMILARITY

11. In quadrilateral $ABCD$, $AD$ is parallel to $BC$, and $AC, BD$ meet at $R$. The line through $R$ parallel to $AD$ meets $AB$ at $P$ and $CD$ at $Q$. Prove that

$$1/PR = 1/AD + 1/BC.$$ 

Deduce that $R$ is the mid-point of $PQ$.

12. $ABCD$ is a parallelogram. $K$ is a point on $CB$ produced so that $AK = AB$. $H$ is a point on $AB$ produced so that $CH = CB$. Prove that $DH = DK$.

13. Consider a regular polygon with 12 sides and vertices $A_1, A_2, \ldots, A_{12}$. Prove that the intersection of the lines $A_1A_5$ and $A_2A_6$ is the same as the intersection of the lines $A_3A_8$ and $A_4A_{11}$. Can you discover any other results of this type for regular polygons? Try the remarkable 18-gon!

14. Given an isosceles triangle $ABC$ with $AB = AC = x$ and $\angle BAC = 20^\circ$. Let $P$ and $Q$, on sides $AB, AC$, respectively, be such that $\angle BCP = 50^\circ$ and $\angle CBQ = 60^\circ$. Show that $\angle QPA$ is independent of the value of $x$ and find the value of this angle.

15. In triangle $ABC$, let $P$ lie on $AC$ so that $AP : PC = 2 : 3$, and let $Q$ lie on $BC$ so that $BQ : QC = 1 : 5$. Let $AQ$ and $BP$ intersect at $T$. Find the ratio $AT : TQ$.

16. $ABCD$ is a parallelogram and $M, N$ are the mid-points of $AB, BC$, $DM, DN$ meet $AC$ at $P$ and $Q$. Show that $AP = PQ = QC$. More generally, suppose that $M$ divides $AB$ in the ratio $t : 1 - t$ and that $N$ divides $CB$ in the ratio $s : 1 - s$ (where $0 < s, t < 1$). Find the ratios $AP : PQ : QC$.

17. Let $m_a$ be the length of the median from $A$ in triangle $ABC$. Show that $4m_a^2 = 2b^2 + 2c^2 - a^2$. 

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18. Show that the medians from $A$ and $B$ in triangle $ABC$ are perpendicular if and only if $a^2 + b^2 = 5c^2$.

19. $ABMC$ is a convex quadrilateral (with the vertices in that order) such that $AB = BC$, $\angle BAM = 30^\circ$ and $\angle ACM = 150^\circ$. Prove that $AM$ bisects the angle $BMC$.

20. Does there exist an equilateral triangle $ABC$ and a point $P$ inside it such that $PA = 3, PB = 4, PC = 5$? If so, what is the length of the side of the triangle? What happens if $P$ is outside the triangle? Is it possible to construct the triangle, given the lengths 3,4,5. Suppose we replace the lengths 3,4,5 by $p, q, r$. Find necessary and sufficient conditions on $p, q, r$ to be able to solve the above problem.

21. $ABCD$ is a rhombus with $\angle ABC = 60^\circ$; $P$ is on $BC$ and $Q$ is on $CD$; one of the angles of triangle $APQ$ is known to be $60^\circ$. Show that triangle $APQ$ is equilateral.

22. Let $ABC$ be a triangle with a right angle at $B$. Draw squares $BCDX, BAEY$ external to the triangle. Prove that $CE, AD$ intersect on the altitude through $B$.

23. A thin rectangular strip of paper is tied into a single knot which is nicely flattened out. The “knot” then gives a regular pentagon. Prove it!

24. Triangle $ABC$ has $AB = AC$ and $O$ lies below $BC$ so that $OA \perp BC$ and $OB \perp BA$. $F$ is any point on the line segment $BC$. A line through $F$ meets $AB, AC$ (extended as needed) at $D, E$ respectively. Show that $FD = FE$ if and only if $OF \perp DE$.

25. $ABC$ is an isosceles triangle with $AB = AC$ and $\angle BAC = 36^\circ$. Show that $AB : BC$ is the Golden Ratio.

26. $M$ is the mid-point of $BC$ in triangle $ABC$; $K$ lies inside the triangle on $AM$ so that $\angle MKB = \angle ABC$. Prove that $\angle CKM = \angle ACB$.

27. A unit square is cut into two rectangles; the smaller rectangle has width $c$. The smaller rectangle can be translated and rotated so that each vertex lies in the interior of exactly one side of the bigger rectangle. Find $c$.

28. $A, B, C$ are three lattice points on a square grid; $\angle ABC = 45^\circ$; $AB, BC$ contain no lattice points except $A, B, C$. Prove that triangle $ABC$ is right-angled.

29. $KLMN$ is a rhombus with $\angle MNK = 120^\circ$; $C$ lies on side $MN$ so that $CN = 2CM$. Find the ratio $\sin \angle CKN : \sin \angle CLM$.

CIRCLE PROBLEMS
30. Given \( \angle UVW = 120^\circ \). \( ABC \) is an equilateral triangle with \( B \) on \( VU \) and \( C \) on \( UW \). What is the locus of the points \( A \)?

31. \( P \) is a point outside a fixed square such that the square subtends an angle \( 45^\circ \) when viewed from \( P \). What is the locus of all such points \( P \)?

32. Equilateral triangle \( ABC \) is inscribed in a circle. \( M, N \) are the midpoints of \( AB, AC \) and \( MN \) extends to meet the circle at \( F \). Show that \( MF : MN = MN : NF \) and find the value of this ratio.

33. A circle has radius 1. Inside half the circle we draw the largest possible disk and in the remaining space we draw the largest possible disk. Find the radii of these disks. Replace the semicircle by a segment cut off by an equilateral triangle inscribed in the original circle and do the analogous construction. What are the radii of the new disks? Do it again using a segment cut off by a square inscribed in the original circle.

34. Two disjoint circles \( C_1, C_2 \) have centers \( A, B \) respectively. The tangents from \( A \) to \( C_2 \) meet \( C_1 \) at \( P, Q \); the tangents from \( B \) to \( C_1 \) meet \( C_2 \) at \( R, S \). Prove that \( PQ = RS \).

35. Five circles lie between two non-parallel lines which touch each circle; the circles touch each other externally. The smallest radius is 4 and the largest radius is 9. What is the radius of the middle circle?

36. Four disks are arranged in a ring so that each touches its two neighbors. Prove that the four points of contact form a cyclic quadrilateral.

37. Four mutually disjoint circles of radius 1 have centers \( A, B, C, D \). Show that \( PA^2 + PB^2 + PC^2 + PD^2 \geq 6 \) for any point \( P \).

38. Three congruent circles are externally mutually touching and are inscribed in a circle \( \Gamma \). \( P \) is any point on \( \Gamma \) and \( PA, PB, PC \) are tangent lines to the three circles. For a suitable labelling of \( A, B, C \) show that \( PA + PB = PC \).

39. \( ABCD \) is a quadrilateral with its four vertices on a circle. \( AB \) and \( CD \) produced meet at \( P \). Given that \( \angle BPC = 60^\circ \), \( AB = 10, CD = 7, BP = 8 \), find the radius of the circle.

40. Inside a circle there are two intersecting circles. One of them touches the big circle at point \( A \), the other touches the big circle at point \( B \). Prove that if segment \( AB \) meets the smaller circles at one of their common points then the sum of the two radii equals the radius of the big circle. Is the converse true?

41. Let \( S \) be the upper half of the circle of radius 1 with its center at \((1,0)\) on the x-axis. Let \( C \) be any small circle centered at the origin. Let \( C \) intersect the y-axis at \( A \) and let \( C \) intersect \( S \) at \( B \). Extend \( AB \) to meet
the positive $x$-axis at $P$. What happens to the point $P$ as the radius of $C$ shrinks to zero?

42. $ABCD$ is a trapezoid inscribed about a circle, $AD$ is parallel to $BC$, and $AB = CD$. The sides $AB, BC, CD, DA$ touch the circle at $P, Q, R, S$. Show that $Q$ is the midpoint of $BC$, $S$ is the midpoint of $AD$, and $PR$ is parallel to $AD$. Let $AD = x, BC = y$. Show that $AB = (x + y)/2(= m$, the arithmetic mean). Show that the distance $g$ between the parallel lines $AD, BC$ satisfies $g^2 = xy$ (so $g$ is the geometric mean). Let $E$ be on $AD$ with $\angle AEB$ a right angle and let $F$ be on $AB$ with $\angle BFE$ a right angle. Show that $(1/x) + (1/y) = 1/BF$ (so that $BF$ is the harmonic mean $h$ of $a$ and $b$). Thus $m > g > h$.

43. Three consecutive angles of a cyclic quadrilateral are in the ratios $2:3:4$. Find these angles.

44. In a circle of radius $R$, chords $AB, CD$ meet at right angles at $P$. Show that $PA^2 + PB^2 + PC^2 + PD^2 = 4R^2$.

45. Three circles have a point in common and they intersect in pairs again at $A, B, C$. Find the sum of the “”angles” of the “circular triangle” $ABC$.

46. Four circles of the same radius intersect in two “triple” points and four “double points” $P, Q, R, S$. Is $PQRS$ a parallelogram?

47. Three circles with radii $r, s, t$ are mutually touching (they do not overlap) and their external common tangents form a triangle. Show that the triangle is equilateral if and only if $r = s = t$. Show that the triangle is isosceles if and only if two of $r, s, t$ are equal. Show that three such circles exist inside any given triangle.

48. A flat TV screen is mounted flush within a rectangular frame. Each side of the TV screen is an arc of a circle whose center is the mid-point of the opposite side of the screen. The diagonals of the TV screen measure 30 cm. and they intersect at an angle of 60 degrees. Find the length of the diagonal of the frame.

49. A triangle $T$ is inscribed in a circle $C$ whose center lies inside $T$. A second triangle $T^*$ is formed by the tangents to $C$ at the vertices of $T$. Show that if $T$ and $T^*$ are similar then they are equilateral. Does this generalize to an n-sided polygon inscribed in $C$?

50. Inside a circle of radius 20 we place a disk of radius 5 that touches the circle. What is the largest square we can fit inside the remaining space inside the circle?

51. $ABCD$ is a quadrilateral with equal diagonals. Show that $ABCD$ is
cyclic if and only if it is an isosceles trapezoid.

52. Circles radius \( R, r \) touch externally at \( P \). A common tangent line touches the circles at \( U, V \) respectively. Prove that \( UV^2 = 4Rr \).

53. \( APB \) is a straight line. Semicircles are drawn on each of the diameters \( AP, PB, AB \) (all on the same side of the line \( APB \)). The common tangent at \( P \) meets the outer semicircle at \( R \). \( AR, BR \) meet the semicircles at \( Q, S \) (respectively). Prove that \( PQRS \) is a rectangle. Prove also that \( QS \) is a common tangent to the smaller semicircles.

54. Let \( ABCD \) be a quadrilateral, let \( AB, DC \) meet at \( E \), and let \( AD, BC \) meet at \( F \). Show that \( ABCD \) is co-cyclic if and only if \( EB + FB = ED + FD \). Show also that \( ABCD \) is co-cyclic if and only if \( EA - FA = EC - FC \).

55. \( ABCD \) is a co-cyclic quadrilateral and the tangent segments from \( A, B, C, D \) to the circle have lengths \( s, t, u, v \) respectively. Show that \( ABCD \) is cyclic if and only if \( su = tv \).

56. \( K \) is the mid-point of chord \( AB \) of a circle. \( PKQ, RKS \) are chords of the circle. \( PS, RQ \) meet \( AB \) at \( U, V \) respectively. Prove that \( K \) is the mid-point of \( UV \).

\section*{AREA PROBLEMS}

57. Can we have a triangle \( S \) whose sides are all \( \leq 1 \), and a triangle \( T \) whose sides are all \( \geq 100 \), but the area of \( S \) is greater than the area of \( T \)?

58. Can we have a triangle \( S \) whose heights (altitudes) are all \( \leq 1 \) but its area is greater than 100? Can we have a triangle \( T \) whose heights are all \( \geq 2 \) but its area is \( \leq 1 \)?

59. \( ABCDE \) is a regular pentagram; \( AC \) meets \( BD \) at \( P \), and \( AD \) meets \( CE \) at \( Q \). What fraction of the pentagram is the area \( BPQE \)? Are there similar results for other stars?

60. Let \( ABCD \) be a convex quadrilateral. Mark \( P, Q \) on \( AB \) so that \( AP = PQ = QB \). Mark \( R, S \) on \( BC \) so that \( BR = RS = SC \). Similarly for \( T, U \) on \( CD \) and \( V, W \) on \( DA \). Draw the lines \( PU, QT, RW, SV \) to form nine quadrilaterals. Show that the innermost quadrilateral has area one ninth of the area of \( ABCD \). Show that all nine little quadrilaterals have the same area if and only if \( ABCD \) is a parallelogram. What if we divide each line segment of the quadrilateral into \( n \) equal pieces? Suppose that \( n \) is even and that we paint the \( n \) little quadrilaterals alternatively black and white; show that the black area is the same as the white area.

61. Three mutually touching (but otherwise non-overlapping) circles have
radius \( r, s, t \). What is the area of the “circular triangle” which they enclose. Draw three common tangents to form the least convex region containing the circles; what is the area of this convex region?

62. Triangle \( ABC \) has \( AB = 9 \) and \( BC : CA = 40 : 41 \). What is the largest area that this triangle can have?

63. \( ABC \) is a right angled triangle. A square is drawn (outwards) on each side of \( ABC \). The picture is made convex by adding three lines to form three new triangles. Prove that each of these three triangles has the same area as \( ABC \).

64. For any triangle \( ABC \) show that \( a^2 = (b - c)^2 + 4\Delta \tan(A/2) \). Show also that \( a^2 + b^2 + c^2 = 4\Delta(cot A + cot B + cot C) \).

65. A flag measuring 9 × 12 has three vertical strips each 3 × 12 with colors red, white and green in succession. Is it possible to cut the flag into four pieces and re-sew it to form a 9 × 12 flag with three horizontal strips each 4 × 9, with colors red, white and green in succession?

66. Let \( ABC \) be any triangle. External to the triangle erect equilateral triangles \( BCP, CAQ, ABR \). Let the centers of these equilateral triangles be \( K, L, M \). Prove that triangle \( KLM \) is equilateral and identify its center. Show also that the lines \( AP, BQ, CR \) are equal in length and that they meet at a point \( F \). (\( F \) is called the Fermat point of the triangle.) Find a formula for the area of triangle \( KLM \) in terms of triangle \( ABC \) and its sides.

67. Let \( ABCD \) be a convex quadrilateral such that each of the triangles \( ABC, BCD, CDA \) each have area 1. What can you conclude about the quadrilateral? Now let \( A_1, A_2, \ldots, A_n \) be a convex \( n \)-gon such that each of the triangles \( A_1A_2A_3, A_2A_3A_4, \ldots, A_{n-1}A_nA_1 \) has area 1. Find the area of the \( n \)-gon for the cases \( n = 5, 6, 7 \).

68. Let \( ABC \) be a triangle and let \( D, E, F \) be points on \( BC, CA, AB \) respectively such that each of the ratios \( BD : DC, CE : EA, AF : FB \) is less than 1. Prove that the area of triangle \( DEF \) is at least one quarter of the area of triangle \( ABC \).

69. A convex quadrilateral is cut along a diagonal; a congruent quadrilateral is cut along the other diagonal. How can these four triangles be put together to form a parallelogram?

70. Triangle \( ABC \) has area 1 and \( a \geq b \geq c \). Prove that \( b \geq \sqrt{2} \).

71. Let \( E, F, G, H \) be the mid-points of sides \( AB, BC, CD, DA \) of convex quadrilateral \( ABCD \), which has area \( S \). Prove that \( S \leq EG.HF \leq (1/4)(AB + CD).(AD + BC) \). Discuss the cases when either or both of the inequalities is equality.
72. An equilateral triangle is covered by five smaller equilateral triangles congruent to each other. Show that it is possible to move the smaller triangles so that we need only four of them to cover all of the big triangle.

73. Given an arbitrary triangle, can we draw a straight line across the triangle that divides the triangle into two pieces of equal area and at the same time divides the perimeter into two pieces of equal length? Consider the same problem when we replace the triangle by an arbitrary oval shaped curve.

74. Given an arbitrary oval shaped region can we cut it into four pieces by two intersecting straight lines so that each of the pieces has the same area?

75. Given two non-overlapping oval regions can we draw one straight line which will simultaneously divide each region into two pieces of equal area?

76. $ABCD$ is a convex quadrilateral and the diagonals intersect at $P$. Prove that $DPAB + DPCD = DPBC + DPDA$. Prove also that the average of the side lengths is $\leq$ the average of the diagonal lengths. What happens to this last result for
   (i) a non-convex quadrilateral
   (ii) a convex polygon?

77. A small square $PQRS$ lies inside a large square $ABCD$ (in general position). Prove that the sum of the areas $ABQP$ and $CDSR$ is the same as the sum of the areas $BCRQ$ and $DAPS$.

78. Let $P$ be an arbitrary point in a rectangle $ABCD$. Prove that the area of $ABCD$ is not greater than $PA.PC + PB.PD$.

79. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called unstuck if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a $6 \times 8$ rectangle, the smallest perimeter has the form $\sqrt{N}$. Find $N$.

80. $ABCDEF$ is a convex hexagon, and each pair of opposite sides is parallel. Prove that $\Delta ACE = \Delta BDF$.

81. Given a triangle, construct a square which has all four of its vertices on the sides of the triangle. Relate the areas of the square and the triangle.

82. $D, E, F$ lie on sides $BC, CA, AB$ so that $BD : DC = CE : EA = AF : FB = 1 : 2$. Show that the triangle formed by $AD, BE, CF$ has area one-seventh of triangle $ABC$. Now draw the other three trisectors from the vertices to enclose a hexagon. Show that the hexagon has area one-tenth of triangle $ABC$. 

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83. Four circles of radius one have their centers located at the points $(1, 0), (0, 1), (-1, 0), (0, -1)$ and lie within the circle center $(0, 0)$ with radius two. Show that the area in the large circle outwith the four small circles is the same as the area of the inner “propeller” formed by the small circles.

84. Three circles of radius $r$ are mutually touching; what is the area trapped between them?

85. $ABCD$ and $AXYZ$ are parallelograms; $X$ lies on $BC$ and $D$ lies on $YZ$. Show that the area of the two parallelograms is the same.

86. Four squares are drawn on the sides of a rhombus, each square external to the rhombus, and these squares have centers $K, L, M, N$. Show that $KLMN$ is a square. Show that the ratio of the area of the square $KLMN$ to the area of the rhombus can take any value greater than 2.

87. In quadrilateral $ABCD$, $E$ is the mid point of $BC$, $F$ is the mid point of $AD$ and $O$ is the mid point of $EF$. Given that $O$ lies on $AC$ prove that triangles $ABC$ and $ADC$ have the same area.

88. In parallelogram $ABCD$, $E$ is the mid point of $AD$ and $F$ is the mid-point of $CD$. $AC$ meets $BE$ at $P$, and $AC$ meets $BF$ at $Q$. Show that $\Delta APE + \Delta BPQ + \Delta CQF = \Delta PAB + \Delta QCB$.

89. In quadrilateral $ABCD$, angles $B$ and $D$ are right angles and $AB = AD$. The perpendicular distance from $B$ to $AD$ is one unit. Find the area of the quadrilateral as a function of $\angle BAD$.

90. A convex pentagon has all its vertices at points with integer coordinates and no three vertices are in a straight line. Show that the area of the pentagon is at least $5/2$.

91. The Saturday Mathematics Club decided to design a flag to wave at club meetings. It had to be square because that is the nicest rectangle. To avoid too simple a design, a point $P$ was chosen at random inside the square. Through the point $P$ we drew two lines parallel to the sides of the square and two lines parallel to the diagonals of the square, because these seemed the most important directions. This divided the square into eight regions. We painted each of these regions alternatively red and white as we turned about the point $P$ (in a counter clockwise direction). There was no good reason for choosing red and white, but prove that we used the same amount of each color in painting the region.

92. A $2 \times 3$ rectangle has vertices at $(0, 0), (2, 0), (0, 3), (2, 3)$. It rotates $90^\circ$ clockwise about the point $(2, 0)$. It then rotates $90^\circ$ clockwise about the point $(5, 0)$, then $90^\circ$ clockwise about the point $(7, 0)$, and finally $90^\circ$ clockwise about the point $(10, 0)$. Find the area of the region above the $x$-
axis and below the curve traced out by the point whose initial position is (1, 1).

93. $ABC$ is an equilateral triangle; $P$ is on $AB$ and $Q$ is on $AC$; $BQ, CP$ intersect at $R$. Given that $APRQ$ has the same area as $RBC$, find $\angle BRC$.

94. Each midline of a convex hexagon bisects the area of the hexagon. Prove that the three midlines are concurrent.

95. We inscribe the largest square in a semicircle. In the remaining space inside the semicircle we inscribe the largest square possible. What is the ratio of the areas of these squares?

96. A line is drawn through the centroid of a triangle. Prove that the area of each piece is at least $4/9$ times the area of the triangle.

97. Rectangle $ABCD$ encloses a semicircle on diameter $AB$; chord $PQ$ of the rectangle is parallel to $AB$ and cuts the semicircle at $X, Y$. Let $\alpha$ be the area of the top segment of the semicircle cut off by $XY$ together with the two circular triangles $APX, BQY$. For which chord is $\alpha$ minimal?

98. $ABCD$ is a convex quadrilateral; $OA', OB', OC', OD'$ are segments that are parallel to and equal in length to $AB, BC, CD, DA$ respectively. What is the ratio of the area of $A'B'C'D'$ to the area of $ABCD$?

99. $ABCD$ is a rectangle and $EFGH, IFJH$ are inscribed rectangles. Show that the area of $ABCD$ is the sum of the areas of the inscribed rectangles.

100. $P$ is any point inside equilateral triangle $ABC$; $D, E, F$ are the reflections of $P$ in the sides of the triangle. Which triangle has the greater area — $ABC$ or $DEF$?

CONCURRENCE, COLLINEARITY AND TRIANGLE PROPERTIES

101. Let $D, E, F$ be the points on the sides $BC, CA, AB$ of triangle $ABC$ so that $D$ is half way around the perimeter from $A$, $E$ half way round from $B$, and $F$ half way round from $C$. Show that $AD, BE, CF$ are concurrent (in the Nagel point).

102. Say that $X, Y$ are a pair of isotomic points for $MN$ if $OX = OY$ where $O$ is the mid-point of $MN$. Let $D, D', E, E', F, F'$ be isotomic points on $BC, CA, AB$ respectively. Show that if $AD, BE, CF$ are concurrent, so are $AD', BE', DF'$. Show that the Gergonne and Nagel points give such an example.

103. Let $AD, BE, CF$ be concurrent and let the circle through $D, E, F$ intersect $BC, CA, AB$ at $P, Q, R$ respectively. Show that $AP, BQ, CR$ are
concurrent.

104. Show that the tangents to the circumcircle of a triangle at the
vertices intersect the opposite sides of the triangle at three collinear points.

105. Let $D, E, F$ lie on $BC, CA, AB$ so that $BD : DC = CE : EA = AF : FB$. Show that triangles $DEF$ and $ABC$ have a common centroid.

106. Let $AD, BE, CF$ be concurrent at $P$. Show that $DE, EF, FD$ intersect the sides $BC, CA, AB$ in three collinear points; the line is called the trilinear polar of $P$ with respect to the triangle $ABC$.

107. Prove that the external bisectors of the angles of a triangle intersect the opposite sides in three collinear points.

108. Prove that two internal bisectors and the external bisector of the
third angle of a triangle meet the opposite sides in three collinear points.

109. Two parallelograms $ABCD$ and $A'B'C'D'$ have a common angle at $C$. Prove that $DD', AB', A'B$ are concurrent.

110. Let $ABCD$ be a parallelogram and $P$ any point. Through $P$ draw
d lines parallel to $BC$ and $AB$ to cut $BA$ and $CD$ in $G, H$, and $AD$ and $BC$ in
$E, F$ respectively. Prove that the diagonal lines $EG, HF, DB$ are concurrent.

111. If the sides $AB, BC, CD, DA$ of quadrilateral $ABCD$ are cut by a

112. Let $P, Q, R$ be points on sides $BC, CA, AB$ of triangle $ABC$. Let 1
be any point on $BC, 2$ the intersection of $1Q$ and $BA, 3$ of $2P$ and $AC, 4$ of
$3R$ and $CB, 5$ of $4Q$ and $BA, 6$ of $5P$ and $AC, 7$ of $6R$ and $CB$. Show that
point 7 coincides with point 1.

113. Show that a triangle with two equal medians is isosceles. Is the same
conclusion true for a triangle with equal altitudes?

114. Show that the feet of the four perpendiculars from a vertex of a
triangle to the four bisectors of the other angles are collinear.

115. The incenter and the three excenters of a triangle determine six line
segments whose mid-points lie on the circumcircle of the triangle.

116. The perimeter of the orthic triangle is $2\Delta / R$.

117. The nine-point circle is tangent to the incircle and to the three
excircles.

118. Any line from the orthocenter to the circumcircle is bisected by the
nine-point circle.

119. In any triangle, $AH^2 + BC^2 = 4AO^2$.

120. In any triangle, $IO^2 = R(R - 2r)$. (Euler)

121. In any triangle, $OI_1^2 + OI_2^2 + OI_3^2 = 12R^2$. 

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122. Let \( D, E, F \) be the mid-points of \( BC, CA, AB \). Let \( O_1, O_2, O_3 \) be the circumcenters of triangles \( ADF, BDE, CEF \); and let \( Q_1, Q_2, Q_3 \) be the corresponding incenters. Show that triangles \( O_1O_2O_3, Q_1Q_2Q_3 \) are congruent.

123. \( D, E, F \) are points on the sides \( BC, CA, AB \) respectively of a triangle \( ABC \). Show that the circles through \( A, E, F; B, D, F; C, D, E \) have a common point \( O \). Identify the point \( O \) when \( D, E, F \) are the mid-points of \( BC, CA, AB \).

124. In triangle \( ABC \), \( D \) is a point on \( BC \) and \( AB = AC \). Triangle \( ABD \) has inradius \( r_1 \). The ex-circle of triangle \( ADC \), away from \( A \), has radius \( r_2 \). Show that, if \( r_1 = r_2 \), then \( r_1 = h/4 \) where \( h \) is the height of the triangle \( ABC \) from \( A \).

125. \( \Gamma \) is a circle inside triangle \( ABC \) that touches \( AB \) and \( AC \). \( BD, CE \) are the other two tangents to \( \Gamma \) and they intersect at \( F \). Prove that the incircle of \( FBC \) and the incircle of \( ABC \) touch \( BC \) at the same point.

126. \( ABCD \) is a square. Congruent circles \( \Gamma_1, \Gamma_2 \) lie inside the square and touch externally. \( \Gamma_1 \) is tangent to \( BC, CD \); \( \Gamma_2 \) is tangent to \( CD, DA \). The other tangents from \( A, B \) to \( \Gamma_1, \Gamma_2 \) meet at \( E \). Prove that the incircle of \( ABE \) is congruent to \( \Gamma_1 \). Is triangle \( ABE \) equilateral?

127. \( A'B'C' \) is a triangle inside equilateral triangle \( ABC \) such that \( AB'A', BC'B', CA'C' \) are straight lines. Prove that each of the four triangles \( ABB', BCC', CAA', A'B'C' \) has the same inradius \( r \). Express \( r \) in terms of side \( BC \). Must triangle \( A'B'C' \) also be equilateral?

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128. Form a sequence of nested golden rectangles (as discussed in the text) \( ABCD, ECDH, LF DH, LNGH, LNOK \), and so on for ever. Show that \( BD \perp CH, AF \perp EG \), and that these four lines meet at a point \( P \), and that each angle at \( P \) is 45°. Show that \( P \) is the unique point common to all the golden rectangles. The construction gives a maximal square inside each golden rectangle. Show that the circle though any such square contains the point \( P \).

129. \( ABCD \) is a cyclic quadrilateral whose diagonals are perpendicular. Show that \( ABCD \) is co-cyclic if and only if at least one diagonal of \( ABCD \) passes through the center of circle \( ABCD \).

130. Make a regular dodecahedron (you need 12 regular pentagons) and a regular isocahedron (you need 20 equilateral triangles). Inside the dodeca-
hedron find a cube whose eight vertices are also vertices of the dodecahedron. How many different cubes can you find and what is the common region of all these cubes? Make a model of this common region and suggest a name for it. Inside the dodecahedron find a regular tetrahedron whose four vertices are also vertices of the dodecahedron. Find five such tetrahedra that have no common vertex and find the region common to these five tetrahedra. Now find another such five tetrahedra! Is there an analogous problem for the icosahedron? There is a duality (vertex-face) between the dodecahedron and the icosahedron; describe a "physical" method to convert each to the other. Find the volume of each of the solids in this problem. Find the angle between adjacent pentagons on the dodecahedron. Find the angle between adjacent triangles on the icosahedron.

131. Three unit squares form a $3 \times 1$ rectangle. The bottom left corner of each square is joined to the top right corner of the rightmost square to form angles $c, b, a$ respectively with the base. Show that $a = b + c$. Suppose now that we have $n$ unit squares in a row instead of just three. Join the bottom left corner of each square to the top right corner of the last square to make angles $a_1, a_2, \ldots, a_n$ with the base. Can we find $n$ so that $a_1 + a_2 + \ldots + a_n$ exceeds 12 right angles? We have $a + 1 + a_2 + a_3$ equal to one right angle. Can the sum of all the $a$’s be an exact number of right angles for any other $n$? Or can some of the $a$’s add up to two right angles? Instead of having one row of unit squares now take two rows; join the bottom left corner of each square on the bottom row to the top right corner of the last square on the top row to form angles $b_1, b_2, \ldots, b_n$. Ask similar questions about these angles.

132. (A glimpse into the different world of affine geometry.) In Euclidean geometry (the usual school geometry!) we say that two triangles are congruent if we can “move” one of them so that it exactly covers the other. The motions that we allow are translations and rotations and reflections. In affine geometry we allow additional motions by (linear)stretching; can you suggest a more precise description of such stretching motions? Prove that any “motion” in affine geometry is of the form $(x, y)$ moves to $(x', y')$ where $x' = a + bx + cy, y' = d + ex + fy$. Given two triangles $T_1$ and $T_2$ prove that there is an affine motion that moves $T_1$ exactly to $T_2$; in other words, there is essentially only one triangle in affine geometry! Under an affine motion, a straight line moves to a straight line (why?). Under an affine motion, distances get changed in general; but equal distances along a fixed line remain equal distances after the motion. Can you see how to use this to prove that
the medians of any triangle meet in one point? What happens to areas under affine motions? Can you produce an affine motion that will change a circle into an ellipse? Since we know the area of a circle is \( \pi r^2 \), can we find the area of an ellipse? In affine geometry, how many “different” parallelograms are there? What about quadrilaterals? Can you think of other theorems in Euclidean geometry that you can prove easily by using affine motions?

133. We are given the points \( O(0, 0, 0), A(1, 0, 0), B(0, 1, 0), C(0, 0, 1) \) in three dimensional space. We want to find another four points \( P, Q, R, S \) so that the eight points give the vertices of a parallelepiped (a parallelepiped is a generalization of a cube in which the six faces are parallelograms instead of squares). How many different parallelepipeds can we get by this process? Can we write down the complete list of other points that occur as a vertex of one of the parallelepipeds? Do all the parallelepipeds have the same volume?

134. Is there a closed 8-sided polygon (not convex!) with exactly one crossover point on each side?

135. Is it possible to construct a configuration of 8 points in the plane so that the perpendicular bisector of any two of the 8 points passes through at least one of the 8 points?

136. We are given ten line segments each of integer length. Is it always possible to find three of these segments which will be the sides of a (single) triangle? What happens in the case that the longest segment is no more than 50 in length?

137. Consider a \( 20 \times 30 \) rectangular grid of unit squares. Is there a straight line which intersects the interiors of exactly 50 of these unit squares?

138. A triangle \( ABC \) is given. How many points \( D \) are there such that the four points \( A, B, C, D \) have a center of symmetry?

139. The angles of a convex pentagon constitute an arithmetic progression. Prove that each angle is at least 36°.

140. John cut a convex paper polyhedron along all of its edges and sent the resulting collection of faces to Jack by mail. Jack pasted the faces together and got a convex polyhedron. Is it necessarily the same polyhedron?

141. \( P \) is an interior point of triangle \( ABC \). Prove that at least one of the angles \( PAB, PBC, PCA \) is no more than 30°.

142. Is there a closed broken line in the plane which intersects each of its segments exactly once and consists of (i) 6 segments (ii) 7 segments?

143. Is it possible to find six points on the plane and join them by non-intersecting straight segments so that each point is joined with exactly (a) three other points (b) four other points?
144. Two mirrors are hinged together to form a double mirror. A ray of light enters and starts to reflect off the two mirrors. Can it continue reflecting inwards for ever or will it reflect out of the mirror angle? What can you say about the number of reflections in terms of the magnitude of the angle?

145. Show that the number of sides of a golygon is divisible by 8. The definition of a "golygon" is: A path with an even number of steps along the integer lattice points of the plane as follows: Start at the origin, move 1 unit north or south, then 2 units east or west, then 3 units north or south, then 4 units east or west and so on. The length of each move always increases by 1, and the direction alternates between horizontal and vertical.

146. On every planet of some planetary system sits an astronomer observing the closest planet. (All distances between planets are different.) Prove that if the number of planets is odd, then at least one of them is not observed.

147. For a set $E$ in 3-space, let $L(E)$ be the set of all points on all lines determined by any two points of $E$. For example, if $T$ consists of the four vertices of a regular tetrahedron, then $L(T)$ consists of the six edges of the tetrahedron, extended infinitely in both directions. We can apply the $L$-operation to $L(T)$. Is it true that every point in 3-space is in $L(L(T))$? Describe all the sets $E$ for which $L(E) = E$.

148. For any triangle, prove that $\frac{1}{2Rr} \geq a^{-2} + b^{-2} + c^{-2}$.

149. A quadrilateral has sides length $a, b, c, d$, diagonals length $m, n$ and opposite angles $\alpha, \gamma$. Prove Bretschneider’s Law of Cosines:

$$m^2n^2 = a^2c^2 + b^2d^2 - 2abcd \cos(\alpha + \gamma).$$

150. Horace the spider is sitting in the middle of the top face of a (solid) cube. He wants to visit the center of each face (always by the shortest route of course!) exactly once and then return to where he started. Such a path is called a Hamiltonian circuit, and it is said to be oriented if we mark the direction (forwards or reverse). How many different Hamiltonian circuits are there (i) oriented (ii) ignoring orientation? Say that two Hamiltonian circuits are equivalent if one transforms into the other via a symmetry of the cube. Are any two Hamiltonian circuits equivalent? Repeat this problem when Horace’s visiting points are the vertices of the cube and he walks along the edges only. Repeat the problem for the regular tetrahedron, octahedron, dodecahedron, and icosahedron.

151. Faces $ABC$ and $BCD$ of tetrahedron $ABCD$ meet at an angle of $30^\circ$. The area of face $ABC$ is 120, the area of face $BCD$ is 80, and $BC = 10$. Find the volume of the tetrahedron.
152. Trapezoid $ABCD$ has sides $AB = 92$, $BC = 50$, $CD = 19$, $DA = 70$ with $AB$ parallel to $CD$. A circle with center $P$ on $AB$ is drawn tangent to $BC$ and to $AD$. Given that $AP = m/n$, find $m + n$.

153. Consider the region $A$ in the complex plane that consists of all points $z$ such that both $z/40$ and $40/z^*$ have real and imaginary parts between 0 and 1 inclusive. (Recall that $z^* = x - iy$ if $z = x + iy$.) What is the area of $A$ correct to the nearest integer?

154. In triangle $ABC$, $P, Q, R$ are on sides $BC, CA, AB$ respectively and $AP, BQ, CR$ meet at $O$. Given that $AO : OP + BO : OQ + CO : OR = 92$ find the value of $(AO : OP) \times (BO : OQ) \times (CO : OR)$.

155. All planets on some planetary system are spheres of unit radius. Mark on each of the planets the set of points that are invisible from any point on any of the other planets. Prove that the sum of the marked areas is equal to the surface area of one planet.

156. Show that it is possible to draw a finite number of non-intersecting circles in the plane so that each of them touches externally exactly $n$ of the other circles, where $n = 3, 4, 5$.

157. $ABC$ is a right triangle, $M$ is the midpoint of the hypotenuse $AC$, and $Q$ is the incenter of the triangle. Given that $AMQ$ is a right angle, find the ratio of the side lengths of the triangle.

158. Given a polygon in the plane let $f$ be the number of faces, $v$ the number of vertices and $e$ the number of edges. Calculate $f, v, e$ for a selection of polygons. Guess an equation between $f, v, e$ and prove it. Does the equation still hold when we allow diagonal lines in the polygons (thus creating new vertices and edges and faces)? Now consider "bent" polygons in 3-space (for example, an open box, or a box with no top or bottom); does the same equation hold?

159. Now consider the above problem for solids; in particular, cube, tetrahedron, pyramids, dodecahedron, icosahedron, stars, etc. What equation now holds for $f, v, e$? Prove it.

160. Why are there only five regular solids? Recall that a regular solid is one in which each face is a congruent regular $k$-gon (with $k$ constant).

161. How many triangles can be formed with sides of integer length when no side exceeds 100 in length? What is the answer if 100 is replaced by $N$?

162. For which regular $n$-gons can all the vertices be given rational coordinates?
163. Show that in any triangle $ABC$ we have
\[ a^2 + b^2 + c^2 \geq 4\sqrt{3} \Delta. \]

164. Does there exist a triangle, two of whose heights are not shorter than the sides on which they are dropped? If such a triangle exists, what are its angles?

165. You are given three squares of sides 2,3,6 respectively. Their total area is 49 square units. There are lots of ways to cut up these three squares into smaller pieces which can then be fitted together to form a square of side 7. What is the least number of pieces that can be assembled to form this $7 \times 7$ square?

166. Prove that, in any triangle,
\[ 3(bc + ca + ab) < (a + b + c)^2 < 4(bc + ca + ab). \]

167. The closed unit disk is $\{(x, y) : x^2 + y^2 \leq 1\}$ and a closed segment consists of two points together with all the points on the line between them. Can I decompose the closed unit disk into an (infinite) collection of closed segments so that each has length $\leq 1/2$ and so that no two of the closed segments ever overlap? Suppose now that I remove a finite number of points from the closed unit disk. Can I solve this problem? Does it matter whether these removed points are inside the unit circle or on it?

168. $P$ is a point on side $AB$ of triangle $ABC$. $Q$ lies on $CA$ with $PQ \parallel CB$, $R$ lies on $CB$ with $QR \parallel BA$, $S$ lies on $BA$ with $RS \parallel AC$. We continue in this fashion to get points $T, U, V$. Prove that $V = P$. Show also that $PU, QR, ST$ are concurrent (at $X$) if and only if $P$ trisects $AB$; in which case, $X$ is the centroid of triangle $ABC$.

169. Stuck at home on a rainy day with no pack of cards with which to play patience, we make up the following game. We place black and white dots on a line according to the following rules:

(i) We start with two dots on the line, a black one with a white one to its right;

(ii) There are two legal moves; (a) we choose a dot and we adjoin two dots of the same color immediately to the left and right of the chosen dot, or (b) we choose a dot that already has dots of the same color on its immediate left and right and we delete these two dots.

Is it possible to end up with exactly two dots on the line, a white one to the left and a black one to its right? Consider other starting and ending configurations.
170. A unit square is partitioned into finitely many squares and we calculate the total perimeter of all the individual squares that are crossed by the dexter diagonal. Can this total perimeter exceed 1993?

171. An ant is trapped inside a triangle. He starts on one side, visits the other two sides and returns to the original point. What is the shortest trip he can make? What if we replace the triangle by a regular polygon? What if we replace the ant with a fly trapped inside a cube and the fly has to visit each face of the cube?

172. I start with the unit circle $C_0$ and then I construct six circles around $C_0$ successively, in clockwise order. $C_1$ has radius 2 and touches $C_0$ and $C_1$. $C_2$ has radius 1/4 and touches $C_0$ and $C_2$. Similarly for $C_4, C_5, C_6$ with radii 1/2, 2, 4 respectively. Prove that $C_6$ also touches $C_1$. Repeat the problem with the six radii given by $a, b, b/a, 1/a, 1/b, a/b$.

173. Let $CH$ be an altitude of triangle $ABC$. Let $R$ and $S$ be the points where the circles inscribed in triangles $ACH$ and $BCH$ are tangent to $CH$. If $AB = 1995, AC = 1994, BC = 1993$, then $RS = m/n$ with $m, n$ co-prime. Find $m+n$.

174. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ with $P, Q, R, S$ on $AB, BC, CD, DA$ respectively. Given $PB = 15, BQ = 20, PR = 30, QS = 40$. Find $m+n$ if the perimeter of $ABCD$ is $m/n$ with $m, n$ co-prime.

175. Three nickels and two dimes are laid in a “circle” so as to touch an inner disk of radius $r$ and also to touch each neighbouring coin. Up to symmetry there are only two ways to do this; find the difference between the inner radii.

176. A hexagon is inscribed in a circle. Five of the sides have length 81 while the sixth, $AB$, has length 31. Find the sum of the lengths of the three diagonals from $A$.

177. I lay a silver dollar on a sheet of paper and trace its boundary. I mark one point on the circle. Is it possible to construct the diametrically opposite point on the circle if the only construction I am allowed is to select two (not coincident) points and draw a silver dollar circle through them?

178. In triangle $ABC$ I draw circle $C_1$ inside the triangle to touch $AB$ and $AC$. I now draw circle $C_2$ to touch $AB$ and $BC$ and circle $C_1$. I now draw circle $C_3$ to touch $BC$ and $CA$ and circle $C_2$. I continue in this fashion around the triangle. Prove that circle $C_7$ coincides with circle $C_1$. Does the conclusion still hold if the three sides of the triangle are replaced by three non-overlapping circles? Consider the analogous problem for a co-cyclic
quadrilateral.

179. The four medians of a tetrahedron are concurrent in a point that quadrisects each median.

180. Find the locus of a point \( P \) the square of whose distance from the hypotenuse of a given right isosceles triangle is the product of its distances from the two legs.

181. Show that the sum of the squares of the medians of a tetrahedron is four-ninths the sum of the squares of the edges.

182. Prove that the six planes through the mid-points of the edges of a tetrahedron and perpendicular to the opposite edges are concurrent in the Monge point of the tetrahedron.

183. Given fixed points \( A, B \) find all points \( P \) such that \( P \) is not the orthocenter of some triangle \( ABC \).

184. Cut an equilateral triangle into three quadrilaterals and a right angled triangle so that the four pieces can be reassembled into a square. Cut a regular octagon into four pentagons and a square so that the five pieces can be reassembled into a square.

185. Four spheres each of diameter 6in. are placed on a table with three mutually touching and the fourth on top. How high are they stacked?

186. Construct inside a triangle three circles which touch each other (externally) and so that each touches two sides of the triangle. It was once thought that this maximized the sum of the areas of three non-overlapping circles inside the triangle; show this claim is false even for an equilateral triangle! What arrangement maximizes the sum of the areas of the three circles?

187. Construct a cyclic quadrilateral given its sides \( a, b, c, d \).

188. Let \( ABCD \) be a quadrilateral, and let \( M, N \) be the mid-points of \( AD, BC \). Show that \( ABCD \) is a trapezoid if \( 2MN = AB + CD \).

189. Find an equilateral triangle whose vertices lie on (i) three parallel lines (ii) three concentric circles.

190. \( AD, BE, CF \) are medians of triangle \( ABC \). \( K \) lies on \( FE \) produced so that \( EK = FE \). Show that if \( AB = AC \), then \( KA = KD \).

191. \( P \) is a fixed point on circle \( C \). Draw all disks with center on \( C \) and boundary circle passing through \( P \). What do we see?

192. Can you cut an arbitrary triangle into three pieces so that the pieces can be rotated and translated (but not flipped) to form the mirror image of the given triangle?
193. Show that any triangle can be dissected by straight cuts into four pieces that can be rearranged to form two triangles similar to the given triangle.

194. $P$ is any point inside triangle $ABC$. Lines are drawn through $P$ parallel to the sides of the triangle so we get three segments formed on each side of the triangle. The side lengths are $a, b, c$ and the corresponding middle segment lengths are $a', b', c'$. Prove that $a : a' + b : b' + c : c' = 1$.

195. Two of the altitudes of a triangle are 9, 29. If the third altitude is also a positive integer, what values can it have?

196. $M$ is the mid-point of $AB$ in triangle $ABC$. The line through $M$ parallel to the angle bisector of $\angle ACB$ meets the triangle again at $N$. Prove that $MN$ bisects the perimeter of triangle $ABC$.

197. Find the shortest line segment that cuts a $3 : 4 : 5$ triangle into two pieces of equal area.

198. Given straight lines $ABFD$ and $AGCE$ with $AB = BC = CD = DE = EF = FG = GA$, find $\angle DAE$. Generalize.

199. Triangle $ABC$ has area 1. The medians $AD, BE, CF$ divide $ABC$ into six triangles. What is the area of the hexagon whose vertices are the centroids of these six triangles?

200. $AB$ is a diameter of a circle $\Gamma$ and $CD$ is a chord of $\Gamma$ with $CD \perp AB$. Triangle $ABC$ has inradius $r$. Inside $\Gamma$, on opposite sides of $CD$ we draw circles tangent to $AB, CD$ and $\Gamma$ with radii $s, t$. Prove that $2r = s + t$. Show, in fact, that the tangents to the incircle of $ABC$ parallel to $CD$ pass through the centers of the other two inscribed circles.

SOME ADDITIONAL PROBLEMS

201. For triangle $ABC$ show that $\angle A = 2\angle B \iff a^2 = b(b + c)$.

202. The rugby place-kicker’s problem. A try is scored in rugby by crossing the endline, say at point $C$. A free kick at the goal is then allowed from any point $P$ on the line though $C$ perpendicular to the endline. Suppose the goal posts are located at $A, B$ on the endline. We wish to choose $P$ to get the easiest kick, i.e. we wish to maximize $\angle APB$. Show that the maximum occurs when $PC^2 = AC \cdot BC$ or equivalently when the bisector of $\angle APB$ makes an angle $\pi/4$ with the endline.

203. Given angle $AOB$ and $P$ within the angle, construct $Q$ on $OA$ and $R$ on $OB$ so that $PQ = PR$ and $\angle QPR = \pi/2$.

204. Given a triangle $ABC$ we say that a function $f(a, b, c, A, B, C)$ is an
invariant if

\[ f(a, b, c, A, B, C) = f(b, c, a, B, C, A) = f(c, a, b, C, A, B); \]

for example, \( bcsinA \) is an invariant \((= 2\Delta)\). Show that the following are invariants:

(i) \( b^2 + c^2 - bc \sin A \)
(ii) \( b^2 + c^2 - 2bc \sin(A + \pi/3) \).

205. [This is called Napoleon’s Theorem, but some scholars doubt if he knew enough geometry to discover it!] Given any triangle \( ABC \), construct equilateral triangles \( PCB, QAC, RAB \) external to the triangle. Show that the circumcircles for these three equilateral triangles have a common point \( F \) inside the original triangle. Show that \( PA, QB, RC \) are concurrent at \( F \) and that each pair intersects at angle \( \pi/3 \). Show also that the centroids of the three equilateral triangles form an equilateral triangle and find a formula for its side length. What happens if we construct the equilateral triangles internal to the given triangle?

206. Circle \( \Gamma \) has radius \( R \) and diameter \( AB \). Circle \( \Gamma_1 \) (resp. \( \Gamma_2 \)) is inside \( \Gamma \), has radius \( s \) (resp. \( t \)) and touches \( \Gamma \) at \( A \) (resp. \( B \)). \( CD \) is the common chord for circles \( \Gamma_1, \Gamma_2 \). In the space between \( \Gamma, \Gamma_1, CD \), draw the largest possible circle \( K_1 \); in the space between \( \Gamma, \Gamma_2, CD \), draw the largest possible circle \( K_2 \). Show that \( K_1, K_2 \) have the same radius, \( r \), and show that \( 1/r = 1/(R - s) + 1/(R - t) \).

207. Let \( ABCD \) be a co-cyclic quadrilateral with corresponding tangential points \( P, Q, R, S \). Show that \( ABCD \) is cyclic if and only if \( PR \perp QS \). Call \( ABCD \) bi-cyclic if it is both cyclic and co-cyclic. Denote \( ABCD \) by \( X \), and the associated \( PQRS \) by \( X^* \). Show that if \( X \) and \( X^* \) are both bi-cyclic, then \( X^{**} \) is a square!

208. Let \( ABC \) be an isosceles triangle with incenter \( I \) and tangential points \( A_1, B_1, C_1 \). Then triangle \( A_1B_1C_1 \) is isosceles with incenter \( I_1 \) and tangential points \( A_2, B_2, C_2 \). Continue in this fashion and identify the limit point of the sequence \( \{I_n\} \). Show also that each angle of the triangle \( A_nB_nC_n \) converges to \( \pi/3 \).

209. Recall that the harmonic mean, \( h \), of \( a \) and \( b \), is defined by the equation \( 2/h = 1/a + 1/b \); it lies between \( a \) and \( b \) and satisfies the ratio equation, \( a : b = (a - h) : (h - b) \). The anti-harmonic mean, \( k \), of \( a \) and \( b \), is defined by the ratio equation \( b : a = (a - k) : (k - b) \). Show that \( k = (a^2 + b^2)/(a + b) \) and that it lies between \( a \) and \( b \), with \( k \geq (a + b)/2 = \)
(h + k)/2. Establish the following geometric construction for k. Let triangle \(ACB\) be right angled at \(C\), with \(a \geq b\), and let \(M\) lie on side \(CB\) so that \(\angle MAB = \pi/4\); show that \(MB\) has length \(k\).

Verify the following appearances of these means in any ellipse. Let the major axis be \(A'A = 2a\) and the minor axis be \(B'B = 2b\). Let the foci be \(F'\) and \(F\) and let \(P'\) and \(P\) lie on the ellipse directly above \(F\) and \(F'\). Then \(FB\) is the arithmetic mean of \(FA'\) and \(FA\), \(FB\) is their harmonic mean and \(FB'\) is their anti-harmonic mean.

210. Two squares of different size intersect in an octogon with vertices denoted by 1, 2, 3, 4, 5, 6, 7, 8. Show that 15 \(\perp\) 37.

211. In triangle \(ABC\), \(a, b, c\) are integers and one altitude is the sum of the other two altitudes. Prove that \(a^2 + b^2 + c^2\) is the square of an integer.

212. Circles \(\Gamma_1, \Gamma_2\) meet at \(A\) and \(B\). The tangent to \(\Gamma_1\) (resp. \(\Gamma_2\)) at \(A\) meets \(\Gamma_2\) (resp. \(\Gamma_1\)) again at \(M\) (resp. \(N\)). \(MB\) meets \(\Gamma_2\) at \(P\), \(NB\) meets \(\Gamma_1\) at \(Q\). Prove that \(MP = NQ\).

213. Let \(ABCD\) be a quadrilateral and let \(M, N\) be the midpoints of \(AB, CD\). If \(MN\) bisects the area of \(ABCD\), show that \(AB \parallel CD\).

214. Let \(K, L, M, N\) be the midpoints of sides \(AB, BC, CD, DA\) of quadrilateral \(ABCD\). Show that \(ABCD\) must be a parallelogram if \(KM + LN\) is equal to the semiperimeter of \(ABCD\).

215. Complete the details of this sketch proof that \(ABC\) is isosceles if the angle bisectors \(L_a, L_b\) at \(A, B\) are equal in length. Suppose that \(a > b\) (so that \(\angle A > \angle B\)). Show \(a/(a + c) > b/(b + c)\). Show \((b + c)L_a = 2bc \sin A/2\), and similarly for \(L_b\). This gives the contradiction \(L_a < L_b\).

216. The four vertices of a regular tetrahedron lie on a sphere of radius \(R\). A sphere of radius \(r\) touches each face of this tetrahedron. Find the ratio \(R/r\).

217. We are given a finite set \(E\) of \(N\) points in 3-space \((N > 3)\) such that any line through two points of \(E\) always contains at least one other point of \(E\). Show that the points of \(E\) all lie on one straight line! Suppose instead that any plane through three non-collinear points of \(E\) always contains at least one other point of \(E\). Must the points of \(E\) all lie on one plane?

218. Can the three altitudes of a triangle be in the ratios 1:2:3?

219. All three vertices of an isosceles triangle have integer coordinates. Is the square of the length of its base always an even number?

220. I cut a corner square out of an 8 \(\times\) 8 chessboard. I am given 21 tiles each of which covers exactly three squares in a line. Can I cover the board exactly with these tiles? What about other boards and other squares cut
221. In how many different ways can a $2 \times n$ checkerboard be covered by $n$ (non-overlapping) dominoes? Same problem for a $3 \times n$ checkerboard with $n$ even.

222. We want to cover a $m \times n$ checkerboard by (non-overlapping) $T$’s. (A $T$ consists of 4 squares, three in a line with one sticking out in the middle.) Show that this is possible if and only if 4 divides each of $m$ and $n$. Now let $m$ and $n$ be odd, and cut out one corner square of the checkerboard; is it ever possible to cover this board with (non-overlapping) $T$’s?

223. Show that, if we can form a triangle with side lengths $a, b, c$, then we can form a triangle with side lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Is the converse true?

224. $ABCD$ is a convex quadrilateral and $AB + BD < AC + CD$. Prove that $AB < AC$.


226. Can two angle bisectors in a triangle be perpendicular?

227. All the points of a circle are arbitrarily painted in one of two colors. Prove that there is an isosceles triangle in the circle whose vertices are all the same color.

228. A square is divided into $n$ equal squares by drawing parallel lines. What is the least number of straight lines needed so that, for each of the $n$ squares, at least one interior point lies on one of the lines? Start working through the cases $n = 1, 2, 3, \ldots$.

229. Is it possible to draw thirteen straight lines across a chess board such that, after cutting it along these lines, each piece will contain at most one center of a square? [The lines themselves must not pass through any center of a square.]

230. In triangle $ABC$, lines $AP, BQ, CR$ are drawn into the triangle. Prove that $AP, BQ, CR$ are concurrent if and only if

$$\sin(PAC) \sin(QBA) \sin(RCB) = \sin(PAB) \sin(QBC) \sin(RCA).$$

231. Rectangle $FMOH$ has $HO = 11$ and $OM = 5$. Triangle $ABC$ is such that $H$ is its orthocenter, $O$ its circumcenter, $M$ is the mid-point of $BC$ and $F$ is the foot of the altitude from $A$. How long is $BC$?

232. Point $M$ lies inside the parallelogram $ABCD$. Given that $\angle MBC = 20, \angle MCB = 50, \angle MDA = 70, \angle MAD = 40$; find the angles of the parallelogram.
233. In triangle \(ABC\), \(AB : BC = k\) and angle \(B\) is not a right angle. Let \(M\) be the mid-point of \(AC\). Lines symmetric to \(BM\) with respect to \(AB\) and \(BC\) meet line \(AC\) at \(D, E\) respectively. Find the ratio \(BD : BE\).

234. \(M\) is a point inside triangle \(ABC\). Prove that \(M\) is the centroid of the triangle if the three triangles \(MAB, MBC, MCA\) are equal in area.

235. In triangle \(ABC\) we are given \(\angle A = 30\) and \(\angle B = 80\). Point \(K\) is chosen inside the triangle so that triangle \(BCK\) is equilateral. Find \(\angle KAC\).

236. In convex quadrilateral \(ABCD\) we are given \(\angle BAC = 25\), \(\angle BCA = 20\), \(\angle BDC = 50\), \(\angle BDA = 40\). Find the acute angle between the diagonals of the quadrilateral.

237. Let \(I\) be the incenter of triangle \(ABC\). Prove that the line \(AI\) passes through the circumcenter of triangle \(IBC\).

238. Let \(I\) be the incenter of triangle \(ABC\) and let the incircle touch \(BC\) at \(D\). Let \(DHI\) be a diameter of the incircle. Prove that the line \(AI\) passes through the excenter \(I_1\).

239. With the usual notation, prove that

\[
aI^A + bI^B + cI^C = \overrightarrow{0}.
\]

Conversely, prove that the incenter is the only point inside the triangle that satisfies this vector equation.

240. Is there a quadrilateral that can be divided by one straight cut into two congruent parts; but neither its diagonals nor the segments connecting the mid-points of opposite sides divide it into equal parts?

241. Let \(d_a, d_b, d_c\) be the distances from \(O\) to sides \(BC, CA, AB\) of triangle \(ABC\). Use the complex number identity

\[
(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = (\beta - \gamma)(\beta + \gamma)^2 + (\gamma - \alpha)(\gamma + \alpha)^2 + (\alpha - \beta)(\alpha + \beta)^2
\]

to deduce that

\[
abc \leq 4(a d_a^2 + b d_b^2 + c d_c^2).
\]

Use this to prove that

\[
\Delta / R^2 \leq \sin A \sin 2A + \sin B \sin 2B + \sin C \sin 2C.
\]

242. Prove that

\[
AB \cdot AM \cdot BM + BC \cdot BM \cdot CM + CA \cdot CM \cdot AM \geq AB \cdot BC \cdot CA
\]

for any four points \(A, B, C, M\) in the plane.
243. Use the complex number identity
\[ \alpha^3(\beta - \gamma) + \beta^3(\gamma - \alpha) + \gamma^3(\alpha - \beta) = -(\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \]
to prove that \( OH \leq R^2/2r \) in any triangle.

244. \( ABCD \) is a trapezoid with \( AD \parallel BC \). Prove that \( ABCD \) is co-cyclic
if and only if \( AD : BC = \cot A/2 \cot B/2 \).

245. Prove that \( ABCD \) is a co-cyclic quadrilateral if and only if the
incircles of triangles \( ABC \) and \( ADC \) touch each other on \( AC \).

246. Prove that the mid-points of the three diagonals of a complete
quadrilateral are collinear (on the Newton-Gauss line). If the quadrilateral
is co-cyclic then the center of the inscribed circle lies on the Newton-Gauss
line (Newton’s Theorem).

247. The four lines of a complete quadrilateral form four triangles. Prove
that the four orthcenters lie on a line perpendicular to the Newton-Gauss
line. Prove that the four circumcircles meet at Miquel’s point \( M \). If four
of the vertices of the complete quadrilateral lie on a circle then \( M \) lies on
the diagonal connecting the other two vertices. The orthocenter is joined to
the circumcenter of each of the four triangles; prove that the perpendicular
bisectors of these segments meet at Harvey’s point.

248. \( P, Q \) are points inside parallelogram \( ABCD \). \( QA, PD \) meet at \( X \),
and \( QB, PC \) meet at \( Y \). Prove that
\[ \Delta PAX + \Delta PBY = \Delta QDX + \Delta QCY. \]

249. Let \( A \) be one of the intersection points of two circles. In each circle,
a diameter is drawn parallel to the tangent to the other circle at \( A \). Prove
that the endpoints of these two diameters form a cyclic quadrilateral.

250. \( ABCD \) is a square and \( M, N \) lie on \( BC, CD \) such that \( BM : MC = CN : ND = 3 : 1 \). Prove that \( AM \perp BN \).

251. Characterize all the triangles that can be cut (by one straight line)
into two isosceles triangles.

252. Two squares (of different sizes) intersect to form an octagon which
is divided into four quadrilaterals by two diagonals of the octagon. Prove
that these diagonals are perpendicular.

253. Two circles intersect at \( A, B \). The tangents to the circles at \( A \) meet
the circles again at \( M, N \). The lines \( BM, CN \) meet the circles again at \( P, Q \)
respectively. Prove that \( MP = NQ \).
254. Side lengths $a, b, c$ are integers and one altitude is the sum of the other two. Prove that $a^2 + b^2 + c^2$ is the square of an integer.

255. $M, N$ are the mid-points of sides $AB, CD$ in quadrilateral $ABCD$. Prove that, if $MN$ bisects the area of the quadrilateral, then $AB \parallel CD$.

256. Prove that the incenter of a triangle lies on the Euler line if and only if the triangle is isosceles.

257. $BK$ bisects the median $AM$ of triangle $ABC$; in what ratio does it divide $AC$?

258. $AEFD$ is a co-cyclic quadrilateral and $B, C$ are the other two vertices of the complete quadrilateral. Prove that the incircles of triangles $ABC$ and $FBC$ touch $BC$ at the same point. Does this latter property characterize co-cyclic quadrilaterals? How is this problem related to Problem 245?

259. Two isosceles right triangles abut at their right angles to form a non-convex quadrilateral. Prove that the mid-points of the sides of this quadrilateral form a square.

260. $ABCD$ is a square, $P, Q$ lie on $AB, CD$ respectively, and $K$ lies on $PQ$. Circles $BPK, DQK$ meet again at $L$. Prove that $L$ lies on $BD$.

261. $ABC$ is an equilateral triangle, $M$ is on $AC$, $N$ is on $AB$, and $AM, BN$ meet at $X$. Given that quadrilateral $AMXN$ has the same area as triangle $BXC$, find the obtuse angle between $AM$ and $BN$.

262. In triangle $ABC$, we have $BA = BC$ and $D$ is the point on $AC$ so the inradius of $BAD$ is the exradius of $BCD$ (external to $DC$). Prove that this common radius is $1/4$ the altitude from $A$.

263. $ABCD$ is a square, $K$ is the mid-point of $AB$, and $L$ lies on $AC$ so that $AL : LC = 3 : 1$. Prove that $\angle KLD$ is a right angle.

264. Each of four sides of a convex pentagon is parallel to one of its diagonals (that has no end points in common with the side). Prove that the same holds for the fifth side too. Does this lead to another proof of Problem 23?

265. Given a rhombus $ABCD$, find the locus of points $M$ such that $\angle AMB + \angle CMD = 180^\circ$.

266. Two (unequal) circles meet at $A, B$. A line through $B$ meets the circles again at $K, M$, and $L$ is the mid-point of $KM$. $E$ and $F$ are the mid-points of the arcs $AK$ and $AM$ respectively (the arcs that omit $B$). Prove that $\angle ELF$ is a right angle.

267. A midline of a polygon is a line joining the mid-points of two sides. In a convex quadrilateral consider three line segments: the two midlines and
one diagonal. Show that, if the other diagonal bisects one of these three line segments then it bisects all three of them.

268. Let $AD, BE, CF$ be the internal bisectors of triangle $ABC$ (with $D, E, F$ on the sides of the triangle). Find $\angle BCA$ given that $\angle ADC = \angle AFE$.

269. Notation is as in the above problem. Given $AB = 21, BC = 20, CA = 28$, find the ratio $\triangle ABC : \triangle AFD$.

270. Chord $AB$ subtends an angle 120 at the center of a circle; $C$ lies on the minor arc $AB$ and $D$ lies on the chord $AB$ such that $AD = 2, BD = 1, DC = \sqrt{2}$. Find the area of triangle $ABC$.

271. In triangle $ABC$, we have $AB = 4, BC = 5, CA = \sqrt{17}$. The point $D$ lies on $AB$ so that $AD = 1$. Find the distance between the circumcenters of triangles $ADC$ and $DBC$.

272. In triangle $ABC$, we have $AB = 5, BC = 2, CA = \sqrt{22}$. A point $D$ is chosen on $AC$ so that $BD = 3$. Find the distance from $D$ to the circumcenter of triangle $ABC$.

273. In triangle $ABC$, we have $AB = 4, BC = 3, CA = 5$. The point $D$ lies on $AB$ so that $DB = 7/8$. The circumcircle of triangle $BCD$ meets $CA$ at $E$. Find $BE$.

274. Four (generic) points in the plane give four triangles and hence four circumcircles. Prove that these four circumcircles have a point in common. Five (generic) points in the plane give five of these configurations of four points and so five intersection points. Prove that these five intersection points lie on a circle. The story continues for six, seven, ... generic points and the general result is called Clifford’s chain theorem.

275. Three (generic) circles in the plane give three pairs of direct common tangents; each pair gives an intersection point. Prove that these three intersection points lie on a line. What happens if we replace the direct common tangents by the transverse common tangents? What happens if we choose any three of these six intersection points?

276. Lines $\ell_1, \ell_2$ are each tangent to four circles $C_1, C_2, C_3, C_4$ (there is one circle in each of the four angles formed) which are in clockwise order. The other direct common tangents to $C_1, C_2$ and to $C_3, C_4$ intersect on $\ell_1$. Prove that the other direct common tangents to $C_1, C_4$ and to $C_2, C_4$ intersect on $\ell_2$. Prove that the radii of the circles satisfy the equation $1/r_1 + 1/r_3 = 1/r_2 + 1/r_4$.

277. The diagonals of quadrilateral $ABCD$ divide the quadrilateral into four triangles with radii (in clockwise order) $r_1, r_2, r_3, r_4$. Prove that $ABCD$
278. For any triangle \(ABC\), prove that there is a circle through \(B, C\) which touches the incircle at \(P\), say. Let the incircle touch \(BC\) at \(T\), and let \(M\) be the mid-point of altitude \(AD\). Prove that \(TM\) passes through \(P\). Now construct points \(Q, R\) analogously. Prove that \(AP, BQ, CR\) are concurrent.

279. \(ABCDEF\) is a hexagon inscribed in a circle center \(O\), and \(AB = CD = EF, XY, Z\) are the mid-points of \(BC, DE, FA\) respectively. Prove that triangle \(XYZ\) is equilateral. The circles centered at \(X, Y, Z\) with radii \(XO, YO, ZO\) respectively meet again in pairs at \(P, Q, R\). Show that \(XYZ\) is the “Napoleon” triangle for triangle \(PQR\) (see Problem 205).

280. Three non-overlapping disks of radii \(a, b, c\) are mutually touching. The largest disk that can be placed in the space between them has radius \(r\). Prove that
\[
\sqrt{f(a)} + \sqrt{f(b)} + \sqrt{f(c)} = 1, \text{ where } f(t) = \frac{r(r + a + b + c - t)}{t(a + b + c)}.
\]
Show also that
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r} = 2(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{r^2}).
\]

281. Quadrilateral \(ABCD\) is both cyclic and co-cyclic. The associated circles have radii \(R, \rho\) and the distance between their centers is \(\delta\). Prove that
\[
\frac{1}{(R + \rho)^2} + \frac{1}{(R - \rho)^2} = \frac{1}{\delta^2}.
\]

282. \(H\) is the orthocenter of triangle \(ABC\). Draw circles on diameters \(AH, BH, CH\) to give three “circular” triangles which are divided into six pieces by \(AH, BH, CH\). List the six pieces in order around \(H\). Prove that the average area of pieces 1,3,5 is the same as the average area of pieces 2,4,6.

283. Given fixed points \(A, B\), describe the set of all (i) circumcenters, (ii) centroids, (iii) incenters, (iv) orthocenters of all possible triangles \(ABC\).

284. \(C\) is a point inside square \(ABDE\) so that triangle \(CDE\) is isosceles with angles of 15 degrees at \(D\) and \(E\). What kind of triangle is \(ABC\)?

285. Show that a triangle is right angles if and only if \(2R + r = s\).

286. Given triangle \(ABC\), for what value of \(x\) is there a point \(P\) so that
\[
PA = x - a, \quad PB = x - b, \quad PC = x - c?
\]

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287. $AB$ is a chord of a circle of radius $r$. $P, Q$ are outside the circle and lie on the line through $A, B$ so that $AP = BQ$. $P, Q$ determine two pairs of tangents to the circle. These four tangents determine four new points of intersection. Find the locus of these points of intersection as $P, Q$ vary.

288. $D$ is the mid-point of side $BC$ of triangle $ABC$, with incenter $I$. $DI$ meets $AB, AC$ at $M, P$ respectively. Find the area of (non-convex) quadrilateral $BMPC$ in terms of $a$ and $\angle A$.

289. $D$ is the mid-point of side $BC$ of triangle $ABC$, with incenter $I$. Given that $ID = IA$, find the smallest possible value of $\angle AID$.

290. Given isosceles triangle $ABC$, prove that there are unique points $A_1, B_1, C_1$ on sides $BC, CA, AB$ respectively so that quadrilaterals $AC_1A_1B_1, BA_1B_1C_1, CB_1C_1A_1$ are co-cyclic. Prove also that the inradius of triangle $ABC$ is twice the inradius of triangle $A_1B_1C_1$.

*Et sequitur, usque ad infinitum . . .!*
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