SECOND ORDER RECURRENCES VIA TWO-BY-TWO MATRICES

Consider the second order recurrence (or difference equation) given by

\[ x_{n+1} = ax_n + bx_{n-1} \]

with \( x_0 = c_0, x_1 = c_1 \). To check various algebraic identities satisfied by the sequence \( x_n \) we can obtain an explicit (but messy) formula for \( x_n \) by first solving the characteristic equation \( \lambda^2 = a\lambda + b \), and then applying the initial conditions to the general solution. The verification of the algebraic identity may involve heavy algebra. But there is a more elegant method which uses \( 2 \times 2 \) matrices; moreover this method suggests which kind of algebraic identities are available.

Write the recurrence as a linear system:

\[
\begin{bmatrix}
  x_{n+1} \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  a & b \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  x_n \\
  x_{n-1}
\end{bmatrix}
\]

or, more briefly, \( u_{n+1} = Au_n \). Notice that

\[
[u_{n+1} \ u_n] = A[u_n \ u_{n-1}].
\]

Iterate this equation (or apply induction) to get

\[
[u_{n+1} \ u_n] = A^{n-1}[u_2 \ u_1].
\]

Let’s begin with the nicest possible case, in which \( b = 1 \) and \( x_0 = 0, \) and \( x_1 = 1 \). Then we get \( A = [u_2 \ u_1] \) and so \( [u_{n+1} \ u_n] = A^n. \) Thus

\[
\begin{bmatrix}
  x_{n+1} \\
  x_n
\end{bmatrix} = A^n
\]

and we immediately get

\[ x_{n+1}x_{n-1} - x_n^2 = \det(A^n) = (\det A)^n = (-1)^n. \]

Notice that this equation holds for \textit{any} choice of \( a \) (not just the Fibonacci recurrence, \( x_{n+1} = x_n + x_{n-1} \)). We also have

\[ A^{2n} = A^nA^n \]
and so
\[
\begin{bmatrix}
x_{2n+1} & x_{2n} \\
x_{2n} & x_{2n-1}
\end{bmatrix} =
\begin{bmatrix}
x_{n+1} & x_n \\
x_n & x_{n-1}
\end{bmatrix}
\begin{bmatrix}
x_{n+1} & x_n \\
x_n & x_{n-1}
\end{bmatrix}.
\]
This gives
\[x_{2n+1} = x_{n+1}^2 + x_n^2 \quad (*\text{)}
\]
and
\[x_{2n} = x_n(x_{n+1} + x_{n-1}) \quad (**\text{)}.
\]
Both of these equations hold for any choice of \(a\).

**EXAMPLE** Given that
\[
\frac{1}{1 - 2x - x^2} = \sum_{0}^{\infty} a_n x^n
\]
prove that
\[a_n^2 + a_{n+1}^2 = a_{2n+2}.
\]

Cross-multiply to get
\[1 = (1 - 2x - x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots).
\]
Compare coefficients on each side to get \(a_0 = 1, a_2 = 2\) and
\[a_{n+1} = 2a_n + a_{n-1} \quad (n \geq 1).
\]
To get the recurrence to hold for \(n = 0\) we need to have \(a_{-1} = 2 - 2 = 0\).
Now let \(x_n = a_{n-1}\) so that we have \(x_0 = 0, x_1 = 1\) and \(x_{n+1} = 2x_n + x_{n-1}\).
From (*\text{)}, with \(n + 1\) in place of \(n\), we get
\[x_{2n+3} = x_{n+2}^2 + x_{n+1}^2
\]
and hence
\[a_{2n+2} = a_{n+1}^2 + a_n^2.
\]

We can make up lots more identities by using other index rules: for example
\[A^{2n} = A^{n+m} A^{n-m}, \quad A^{(k+1)n} = A^n A^{kn}
\]
and so on.
Now we start to allow more general cases. Suppose that we still have $b = 1$ but we have arbitrary starting values $c_0$ and $c_1$. With $c_2 = ac_1 + c_0$ we get
\[
\begin{bmatrix}
  x_{n+1} & x_n \\
  x_n & x_{n-1}
\end{bmatrix} = A^{n-1} \begin{bmatrix}
  c_2 & c_1 \\
  c_1 & c_0
\end{bmatrix}
\]
and hence
\[
x_{n+1}x_{n-1} - x_n^2 = (-1)^{n-1} (c_2c_0 - c_1^2).
\]
Notice that we can have $c_2c_0 - c_1^2 = 0$ when the point $(c_0, c_1)$ lies on the conic $x^2 + axy - y^2 = 0$. This conic is in fact a pair of straight lines since it amounts to $(x + \frac{a}{2}y)^2 = (1 + \frac{a^2}{4})y^2$.

We also have
\[
[u_{2n+1} \ u_{2n}] = A^n A^{n-1}[u_2 \ u_1] = A^n[u_{n+1} \ u_n].
\]
When $c_2c_0 - c_1^2 \neq 0$, the matrix $[u_2 \ u_1]$ has inverse matrix, say $V$, and we get
\[
\begin{bmatrix}
  x_{2n+1} & x_{2n} \\
  x_{2n} & x_{2n-1}
\end{bmatrix} = A \begin{bmatrix}
  x_{n+1} & x_n \\
  x_n & x_{n-1}
\end{bmatrix} V \begin{bmatrix}
  x_{n+1} & x_n \\
  x_n & x_{n-1}
\end{bmatrix}.
\] (#)
This gives $x_{2n+1}$ and $x_{2n}$ as quadratic forms in the variables $x_{n+1}, x_n, x_{n-1}$.

What happens in the degenerate case when $c_2c_0 = c_1^2$?

Finally we drop the requirement that $b = 1$. Then we get $\det A = -b$ and so
\[
x_{n+1}x_{n-1} - x_n^2 = (-b)^{n-1}(c_2c_0 - c_1^2).
\]
We again get the equation (#), but now the formulas are a little more complicated since we have the additional parameter $b$.

We can generalize all this for higher order recurrences, but the formulas are much more complicated, especially the determinantal ones. The matrix $A$ which appears is essentially the rational canonical form that appears in advanced linear algebra.