STIRLING’S FORMULA

Stirling’s formula states that

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \]

The symbol \( \sim \) is read as is asymptotic to and it means that the quotient of the two formulas converges to 1 as \( n \to \infty \).

The key idea of the proof (other than the mysterious constant \( \sqrt{2\pi} \)) is to estimate the integral of \( \log t \) by the mid-point rule. The simple (and elegant) argument below does this without our having to know anything about the error formula for the mid-point rule for numerical integration. By elementary calculus, for \( \nu = 1, 2, \ldots, n \), we get

\[
\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt = \int_0^{\frac{1}{2}} \log(\nu + t) + \log(\nu - t) \, dt = \int_0^{\frac{1}{2}} \log \nu^2 + \log(1 - \frac{t^2}{\nu^2}) \, dt.
\]

The convexity of \( \log \) shows that \( -\log(1 - x) < cx \) for \( 0 < x < \frac{1}{2} \) and so it follows that

\[
\int_0^{\frac{1}{2}} \log(1 - \frac{t^2}{\nu^2}) \, dt = -C_\nu
\]

where \( C_\nu = O(1/\nu^2) \). We thus have

\[
\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t \, dt = \log \nu - C_\nu.
\]

Sum from \( \nu = 1 \) to \( \nu = n \) to get

\[
\log n! = \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log t \, dt + \sum_{\nu=1}^{\nu=n} C_\nu.
\]

Since \( C_\nu = O(1/\nu^2) \) we get \( \sum_{\nu=1}^{\nu=n} C_\nu = \alpha + \epsilon_n \) where \( \epsilon_n \to 0 \) as \( n \to \infty \). Elementary integration now gives

\[
\log n! = (n + \frac{1}{2}) \log(n + \frac{1}{2}) - \frac{1}{2} \log \frac{1}{2} - n + a + \epsilon_n.
\]

Since

\[
\log(n + \frac{1}{2}) - \log n = \log(1 + \frac{1}{2n}) = \frac{1}{2n} + O(1/n^2)
\]
it follows that
\[ \log n! = (n + \frac{1}{2}) \log n - n + b + \delta_n \]
where \( \delta_n \to 0 \) as \( n \to \infty \). Now exponentiate to get
\[ n! = b n^{n+\frac{1}{2}} e^{-n} e^{\delta_n}. \]

It remains to prove that \( b = \sqrt{2\pi} \).

Let
\[ u_n = \frac{n!}{\sqrt{n}} \left(\frac{e}{n}\right)^n. \]

Then \( u_n \) converges to \( b \) and so \( u_{2n} \) also converges to \( b \) and \( u_n^2 \) converges to \( b^2 \). It follows that \( u_n^2 / u_{2n} \) converges to \( b^2 / b = b \). This gives
\[ b = \lim_{n} \frac{(n!)^2}{n} \left(\frac{\pi}{2}\right)^{2n} = \lim_{n} \frac{2^{2n}(n!)^2 \sqrt{2}}{(2n)! \sqrt{n}} \]
and hence
\[ b^2 = \lim_{n} \left( \frac{2n(2n-2) \cdots 2}{(2n-1)(2n-3) \cdots 1} \right)^2 \frac{2}{n}. \]

If we know the Wallis product, then the proof is complete. Otherwise we recall the following formulas from elementary calculus (derived by integration by parts and reduction formulas):
\[ s_{2n} = \int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{2n-1}{2n-2} \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \]
\[ s_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} \theta \, d\theta = \frac{2n}{2n+1} \frac{2n-2}{2n+1} \cdots \frac{2}{3}. \]

Since
\[ \frac{2n-1}{2n} S_{2n} = S_{2n+2} \leq S_{2n+1} \leq S_{2n} \]
it follows that \( S_{2n+1} / S_{2n} \) converges to 1. This leads to \( b^2 = 4\pi / 2 \) and hence \( b = \sqrt{2\pi} \) as required.