TRANSPORT, MIXING AND FLUIDS

ANNA MAZZUCATO

PENN STATE UNIVERSITY

48th Annual Spring Lecture Series

Department of Mathematics

UNIVERSITY OF ARKANSAS

Fayetteville, AK, May 5-7, 2023

Partially supported by the National Science Foundation

Charlie

In memory of Charles Doering (1956 - 2021)

friend and mentor

(Photo courtesy of Center for Complex Systems, University of Michigan)

0. INTRODUCTION

Study how well a passive scalar tracer (e.g. dye in water) can be mixed by incompressible flows.

Important problem:

① Analytically, related to irregular transport, anomalous and enhanced dissipation (=> turbulence)

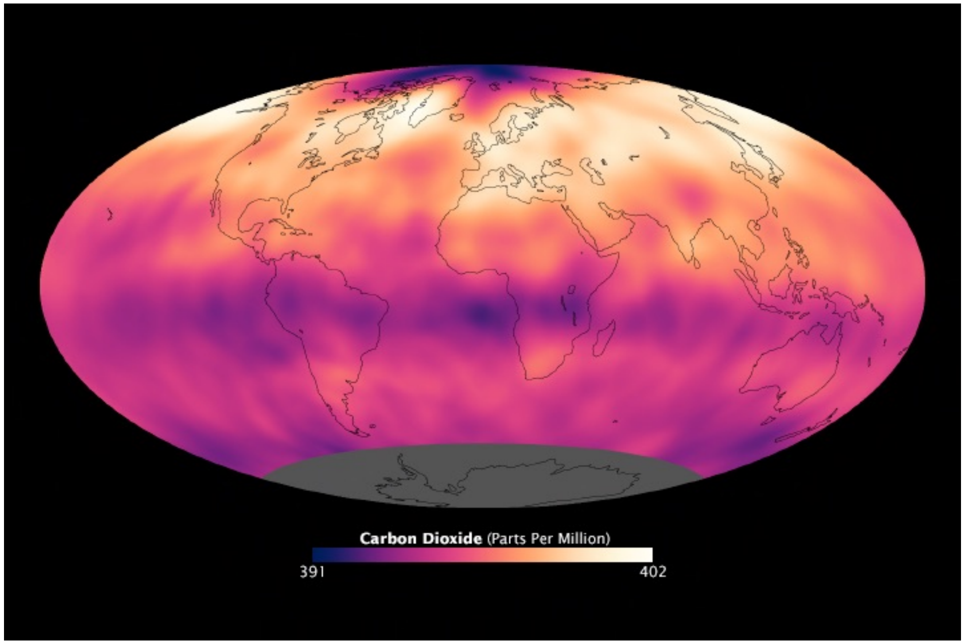
② In applications, e.g. pollutant contamination, CO_2 dispersal, efficient combustion, extrusion in manufacturing.

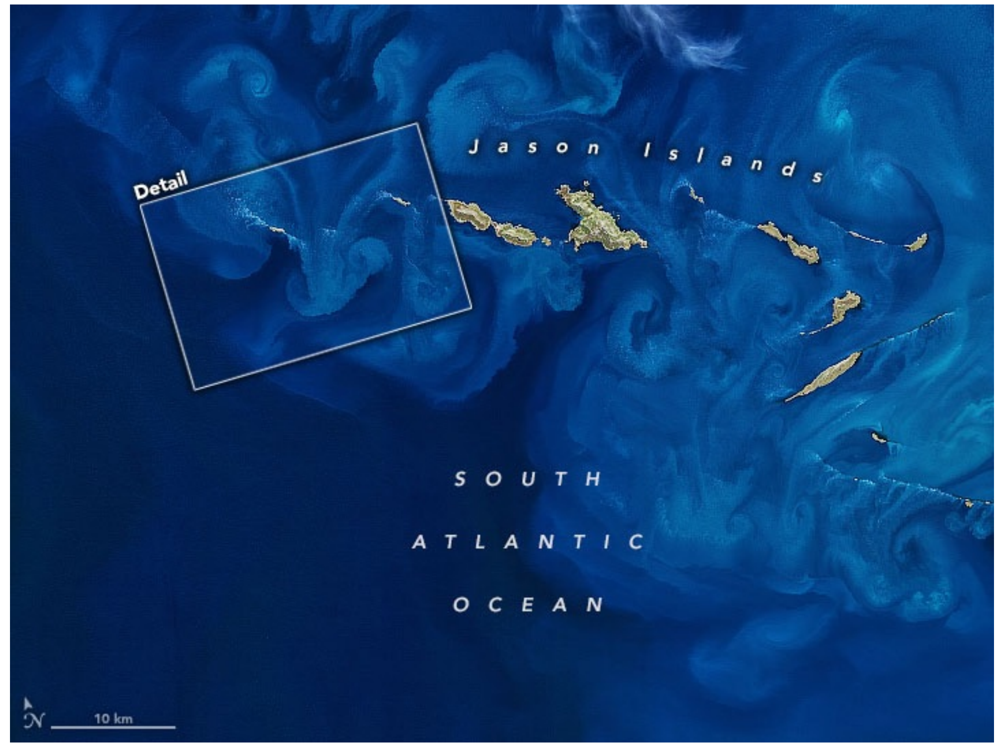
Two main mechanisms for mixing (Danckwerts, Eckart, Welander '50s)

- filamentation due to transport by volume-preserving flows (stirring) => growth of derivatives of tracer.
- diffusion.

We will primarily concentrate on effect of stirring and neglect diffusion, sources and sinks.

MIXING IN THE OCEAN AND ATMOSPHERE





global CO_2 concentration in 2013 (record year)

active mixing and churning of ocean waters

(courtesy of NASA Visible Earth)

RELATED WORKS

Large literature on mixing:

- turbulence (Boffetta et Al., Gotoh-Watanabe)
- ergodic theory (Aref, Liverani, Ottino, Dolgopyat...)
- homogenization, singular perturbation (Otto, ...)
- optimal control (Couffied, Hu-Wu)

In incompressible fluid mechanics, connection with:

- Relaxation (dissipation) enhancing flows (Constantin et Al.,..)
- Inviscid damping and stability of Euler flows (Bedrossian-Masmoudi, Bedrossian-Coti Zelati, ...)

Our approach is based on tools from PDEs and geometry (classical geometry and geometric analysis).

I. IRREGULAR TRANSPORT

Passive scalar assumed to solve a linear transport equation:

$$\partial_t \theta + u \cdot \nabla \theta = 0 \quad , \quad \theta(0) = \theta_0, \qquad (T)$$

where $\theta : \Omega \times [0,T] \to \mathbb{R}$, $u : \Omega \times [0,T] \to \mathbb{R}^d$, $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, $d \geq 2$, u given, $\operatorname{div} u = 0$.

Assume u has limited (Sobolev) regularity. Even when u is regular, dependence of θ on the flow of u is nonlinear.

Because u is divergence free, (T) is (formally) equivalent to a continuity equation:

$$\partial_t \theta + \operatorname{div}(u\theta) = 0, \quad \theta(0) = \theta_0. \qquad (C)$$

For most lectures $\Omega = \mathbb{T}^2$. Refer to u as the advecting velocity.

Lipschitz-continuous velocity

When $u \in L^\infty([0,T], \mathrm{Lip}(\mathbb{T}^2))$, the classical Cauchy-Lipschitz theory applies $\Rightarrow$ solve (T) by the Method of characterstics.

(a) Any weak solution θ of (T) with $\theta_0 \in L^p(\mathbb{T}^2)$, $1 \leq p \leq \infty$, is a Lagrangian solution:

$$\boxed{\theta(x,t) = \theta_0\left(\Phi^{-1}(x,t)\right)}$$

with Φ the flow of u:

$$\begin{cases} \frac{d}{dt}\Phi(x,t) = u(t, \Phi(x,t)), \\ \Phi(x,0) = x. \end{cases}$$

Φ^{-1} referred to as the back-to-labels map.

(b) the flow of u, Φ, is also Lipschitz in space, and the norm bound holds:

$$\boxed{\|\nabla_x \Phi(\cdot,t)\|_{L^\infty(\mathbb{T}^2)} \leq e^{Lt}}$$

where L is the Lipschitz constant of u.

(c) Weak solutions of (T) are unique and θ is Lipschitz continuous if θ_0 is.

Sobolev velocities

We now assume $u \in L^\infty([0,T]; W^{1,p}(\mathbb{T}^2))$, $1 \leq p < \infty$.

Notation: We denote the L^p-based Sobolev spaces, as usual

$$W^{k,p}(\mathbb{T}^2) = \{ f \in L^p(\mathbb{T}^2) \,/\, \nabla^k f \in L^p(\mathbb{T}^2)\}, 1 \leq p \leq \infty.$$

the Lipschitz space $Lip = W^{1,\infty}$.

If $\theta_0 \in L^p(\mathbb{T}^2)$, weak solutions still exist (provided $u \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$) but they may not be unique.

Note that if $p \geq 2$, $u \in W^{1,p} \Rightarrow u \in L^q$, $\forall q < \infty$, by Sobolev embedding.

Uniqueness can be restored for renormalized solutions (DiPerna-Lions '80s) $\Rightarrow$ informally, θ is a renormalized solution if $\beta(\theta)$ is a weak solution of (T) for all functions $\beta \in C^1_b(\mathbb{R})$, $\beta(0) = 0$.

Remark: if $u \notin L^q(\mathbb{T}^2)$, then non uniqueness for Lagrangian solutions with Sobolev velocities was recently obtained by convex integration (Székelyhidi – Modena '19, Cheskydov - Luo '20).

Properties of renormalized solutions:

1. Weak solutions obtained by mollification and by vanishing viscosity (add $\varepsilon \Delta \theta$ and send $\varepsilon \to 0$) are renormalized if $\theta_0 \in L^p$, $p \geq 1$ (DiPerna-Lions '80s, LeBris-Lions '04, Crippa-Spirito '15)

2. If θ is renormalized, the L^p norm of θ is conserved by the flow:

$$\boxed{\| \theta(t,\cdot) \|_{L^p} = \| \theta_0(\cdot) \|_{L^p}} \quad \forall t \in [0,T]$$

if u is divergence free.

3. The theory of renormalized solutions, in particular uniqueness, can be extended to velocity fields $u \in L^1([0,T]; BV(\mathbb{T}^2))$ with θ_0 bounded, where BV is the space of functions with Bounded variation (weak closure of $W^{1,1}$) $\Rightarrow$ ∇u is a measure (Ambrosio '90s).

The result in 3 is sharp (a counterexample discussed later).

II. MIXING NORMS AND RATES

Informally, scalar θ is perfectly mixed if $\theta = \bar{\theta}$, where $\bar{\theta}$ is the average of θ: $\boxed{\bar{\theta}(t) := \fint_{\mathbb{T}^2} \theta(x,t)\,dx.}$

Assumption: throughout assume $\theta_0 \in L^\infty(\mathbb{T}^2)$, $\bar{\theta} = 0$.

Because we work with weak solutions, $\bar{\theta}(t) = 0 \;\forall t$ if $\bar{\theta}_0 = 0$.

Definition: $\theta(t)$ is perfectly mixed if $\theta(t) \xrightarrow[t \to T_{mix}]{} 0$ weakly in $L^2(\mathbb{T}^2)$. $T_{mix} \leq \infty$ is called the mixing time.

Note that θ cannot converge to zero strongly, as the L^2 norm of θ is conserved.

Ergodic mixing (strong): the flow of u is mixing if, for any two Borel measurable sets A, B with positive measure

$$\boxed{m(\Phi^*_{t_n}(A) \cap B) \xrightarrow[n \to \infty]{} m(A)\, m(B)} \qquad (EM)$$

(m Lebesgue measure, Φ^*_t push-forward).

Condition (EM) says that θ_0 and $\theta(t)$, when $\theta_0 = \chi_A$, decorrelate as $n \to \infty$. Using that simple functions are dense if Φ is mixing in the ergodic sense, then any $\theta_0 \in L^\infty(\mathbb{T}^2)$ is perfectly mixed at the mixing time.

Two fundamental questions arise:

① Given the regularity of u, what is the optimal mixing rate?

② Given the regularity of u, is the mixing time T_{mix} finite or infinite?

To answer ① and ②, introduce quantitative measures of mixing.

Negative Sobolev norms: for convenience, use L^2- based norms $\Rightarrow$ defined using Fourier series. Let f be a distributions on $\mathbb{T}^2$ and let $\langle f, e_k \rangle =: \hat{f}_k$, $k \in \mathbb{Z}^2$, $e_k(x) = e^{-ik\cdot x}$, be the k-th Fourier coefficient of f. For $s \in \mathbb{R}$, define the s-norm:

$$\boxed{\|f\|_s = \|f\|_{H^s} := \Big(\sum_{k \in \mathbb{Z}^2, k \neq 0} |k|^{2s} |\hat{f}_k|^2 \Big)^{1/2}}$$

Mix-norms

Using rescaling, one can see that the s-norm amplifies large scales and penalizes small scales if $s<0$.

Mixing arises from the creation of small (space) scales by the flow $\Rightarrow$ negative Sobolev norms of $\theta(t)$ will decay in time.

Lemma (Doering–Thiffeault '11): $\{\theta_n\} \subset L^2(\mathbb{T}^2)$, $\bar{\theta}_n = 0$.

$$\theta_n \xrightarrow[n\to\infty]{} 0 \iff \|\theta_n\|_s \xrightarrow[n\to\infty]{} 0, \quad s<0$$

We can use any negative Sobolev norm to quantify mixing.
Refer to negative Sobolev norms as mix-norms.

In 2D, normalizing $\|\theta_0\|_{L^2} = 1$, the -1 norm has the dimension of a length scale.

Definition: $\boxed{\mathcal{E}_f(\theta)(t) := \|\theta(t)\|_{-1}}$ is the functional mixing scale for scalar θ at time t.

Other mix-norms used in literature ($s=-\frac{1}{2}$, Mathews-Mezic-Petzold).

Related geometric concept, the characteristic length scale of tracer at t.

ε-mixing and rearrangement cost

Definition: A measurable set A with $m(A)=\frac{1}{2}m(\mathbb{T}^2)$, is κ-mixed to scale ε, when $0<\kappa<1/2$, $\varepsilon>0$, if for any $x\in\mathbb{T}^2$:

$$m(D(x,\varepsilon)) \leq m(D(x,\varepsilon)\cap A) \leq (1-\kappa)\, m(D(x,\varepsilon)) \qquad (*)$$

where $D(x,\varepsilon)$ is the disk centered at x with radius ε.

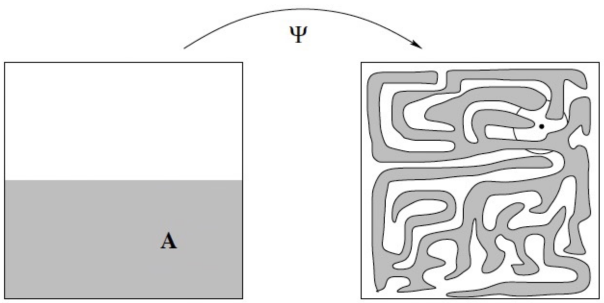
Apply this notion to the level sets of θ_0. Assume for simplicity θ_0 is a binary function $\theta_0=\begin{cases}1 & \text{on } A_0\\ -1 & \text{on } A_0^c\end{cases}$, $m(A_0)=\frac{1}{2}m(\mathbb{T}^2)$.
Set $A_t := \Phi_t^*(A)$.

Definition: $\boxed{\varepsilon_g(t) := \operatorname{Inf}\{\varepsilon>0 \,/\, (*) \text{ holds for } A=A_t\}}$ is the geometric mixing scale at time t.

Fact: $\mathcal{E}_g$ and $\mathcal{E}_f$ are not equivalent (Lin-Lunasin-Novikov-M.-Doering '12). But if flow of u Φ is mixing, the $\mathcal{E}_f$ and $\mathcal{E}_g$ decay at similar rates (up to constants) $\Rightarrow$ $\mathcal{E}_f = \mathcal{E}_g = 0$ at $t = T_{mix}$

Conjecture (cost of rearranging set, A. Bressan): Let Φ be the flow at time 1 of a (sufficiently regular) vector field u. If $\Phi(A)$ is mixed to scale ε, then $\exists\, C = C(A, K)$ such that

$$\boxed{\int_0^t \int_{\mathbb{T}^2} |\nabla u(x,\tau)|\, dx\, d\tau > C\, |\log \varepsilon|} \quad \text{(BC)}$$



Conjecture is still open. Proved if ∇u replaced by $|\nabla u|^p$, $p>1$. (Crippa-De Lellis, '08).

Proof uses the following quantitative estimates for so-called regular Lagrangian flows:

$$\boxed{\int_{D(x,r)\cap G_\lambda} \log\left(\frac{|\Phi(x,t) - \Phi(y,t)|}{r} + 1\right) dy \lesssim C_\lambda \int_{\mathbb{T}^2} |\nabla u(x,t)|^p\, dx} \quad \text{(L)}$$

where $G_\lambda := \{ |\Phi(x,t)| \leq \lambda,\ \forall\, t \in [0,T] \}$.

Estimate (L) can be viewed as an integrated form of the classical Cauchy-Lipschitz estimate:

$$\boxed{\log\left(|\Phi(x,t)-\Phi(y,t)|+1\right) \lesssim C\,\|\nabla u(t)\|_{L^\infty}.} \qquad (CL)$$

Incompressible flows with Sobolev regularity are regular Lagrangian.

Mixing Rates

If θ_0 is mixed by u, both $\varepsilon_f, \varepsilon_g$ will decay to 0.

How fast the mixing scales decays and whether perfect mixing is achieved in finite or infinite time depend on u and possibly θ_0.

Estimates (L) and (CL) indicates that ∇u is key in controlling the trajectories $\Rightarrow$ we distinguish 3 cases:

(a) $u \in L^\infty([0,\infty); W^{s,p}(\mathbb{T}^2))$, for some $0 \leq s < 1$, $1 \leq p \leq \infty$;

(b) $u \in L^\infty([0,\infty); W^{1,p}(\mathbb{T}^2))$, for some $1 \leq p \leq \infty$;

(c) $u \in L^\infty([0,\infty); W^{s,p}(\mathbb{T}^2))$, for some $s > 1$, $1 \leq p \leq \infty$.

If u is the velocity of a physical fluid flow, then:

(a) includes the case of energy constrained flows (energy $\|u(t)\|^2_{L^2}$);

(b) includes the case of enstrophy constrained flows (enstrophy $\|\omega\|^2_{L^2} = \|\nabla u\|^2_{L^2}$, $\omega = \operatorname{curl} u$ vorticity);

(c) includes the case of palinstrophy constrained flows (palinstrophy $\|\nabla \omega\|^2_{L^2}$, $\omega = \operatorname{curl} u$ vorticity).

Optimal rates are known in all three cases now, using both deterministic and stochastic flows.

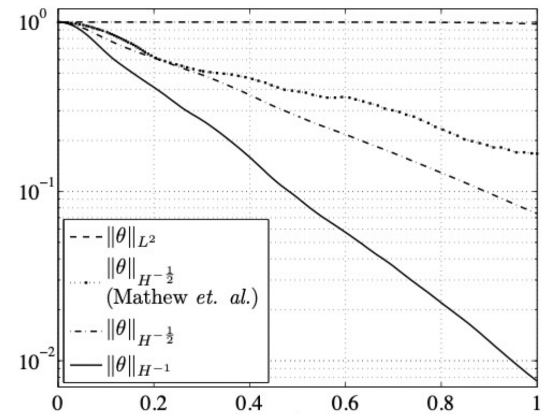
We focus the discussion on decay of functional scale $\|\theta(t)\|_{-1}$.

Numerical simulations support:

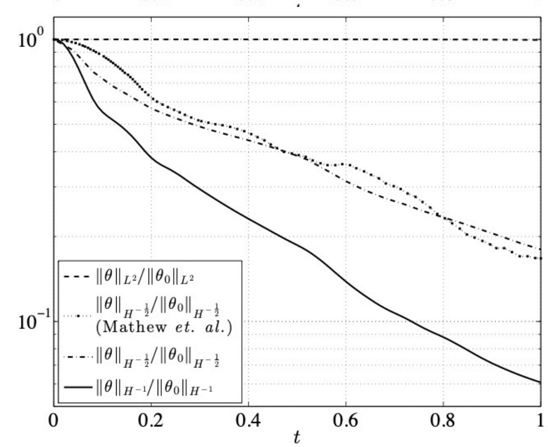
(a) finite-time perfect mixing;

(b) exponential infinite-time mixing;

(c) exponential infinite-time mixing.



(b)



(c)

III. OPTIMAL MIXING

Finite-time mixing

Since (T) is time-reversible and $\theta \equiv 0$ is always a solution, finite-time mixing is only possible if non uniqueness of (weak) solutions holds.

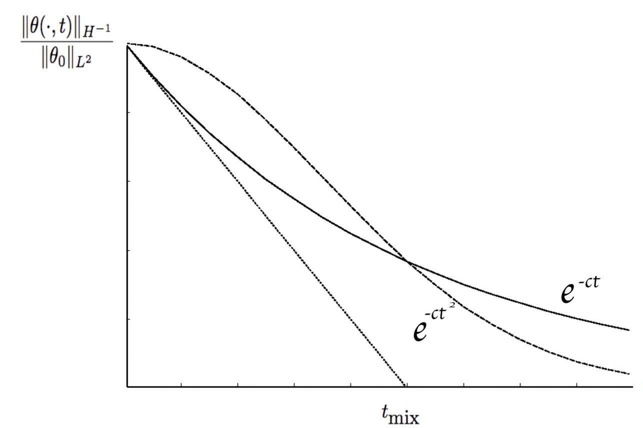
$\Rightarrow$ by the DiPerna–Lions–Ambrosio theory, impossible under enstrophy
∘ palinstrophy constraint.

Under an energy budget ($\|u(t)\|^2_{L^2} \leq C \;\; \forall t$), finite-time mixing is consistent with lower bound on the mixing scale, obtained by simple energy estimates: write $\theta = \Delta \phi$, potential $\phi \in H^2 \Rightarrow$
$\|\nabla \phi\|_{L^2} = \|\theta\|_{H^{-1}}$ $\Rightarrow$ integrating by parts:

$$\frac{d}{dt} \|\theta(t)\|_{H^{-1}} \geq - \|u(t)\|_{L^2} \|\theta(t)\|_{L^2} = - \|u(t)\|_{L^2} \|\theta_0\|_{L^2}$$

$\Rightarrow$ linear lower bound.

Construct a simple example that achieves finite-time mixing for our initial condition.



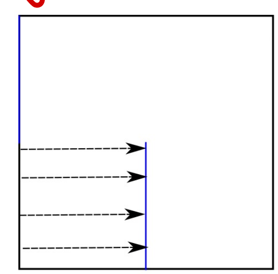
Optimal mixer (energy budget)

Already implicitly present in work of Bressan & DePauw.

Set $A_0 = \frac{1}{2}$ torus, $\theta_0(x) = \begin{cases} 1 & \text{on } A_0, \\ -1 & \text{on } A_0^c. \end{cases}$

Employ a "slice and dice" strategy: apply piecewise constants shear flows, alternating vertical with horizontal, halving time at each step

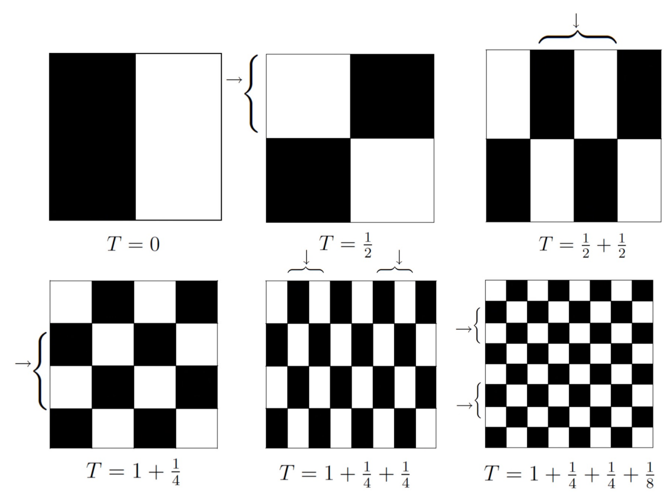
Ex: horizontal shear $u = \begin{cases} 0, & 1/2 < x < 1, \\ (u_1(y), 0) & 0 < y < \frac{1}{2}. \end{cases}$



Use scaling properties of -1 norms:

$$f_\lambda(x) := f(\lambda x) \Rightarrow \|f_\lambda\|_{-1} \le \lambda^{-1} \|f\|_{-1}$$

$\Rightarrow$ mixing scale decrease by a factor of $\frac{1}{2}$ at each iteration.



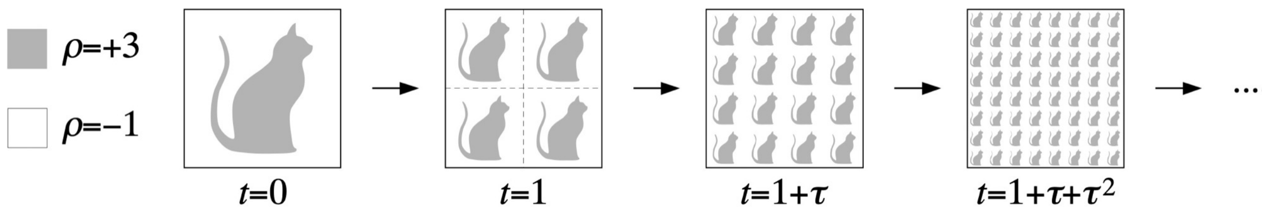
Perfect mixing at time $T_{mix} = \sum_{n \in \mathbb{N}} 2^{-n} = 2$.

Self - similar mixing

Previous construction is an example of self - similar mixing

$\Rightarrow \exists\, t_n,\ n \in \mathbb{N}$, such that $\boxed{\theta(x, t_n) = \theta(Nx, t_{n+1}),\ N \in \mathbb{N}}$,

that is, tracer field at time t_n consists of exact replicas of the field patterns at time t_{n-1} at smaller scales.



Using rescaling easy to show:

(a) $u \in L^\infty([0,\infty); W^{s,p})$, $0 \le s < 1$ $\Rightarrow$ finite - time perfect mixing

(b) $u \in L^\infty([0,\infty); W^{s,p})$, $s = 1$ $\Rightarrow$ exponential - in - time mixing

(c) $u \in L^\infty([0,\infty); W^{s,p})$, $s > 1$ $\Rightarrow$ polynomial - in - time mixing

For (a), (b) self - similar mixing is optimal. For (c), suboptimal.

Exponential mixing

Definition: u mixes θ_0 exponentially in time if there exists constants $C, c > 0$ (depending on u and possibly θ_0) such that

$$\mathcal{E}_f(\theta)(t) = \|\theta(t)\|_{-1} \leq C e^{-ct}, \quad \forall\, t > 0.$$

From Bressan's conjecture, expect an exponential lower bound on $\mathcal{E}_f(\theta)$ if $u \in L^1([0,T]; W^{1,p}(\mathbb{T}^2))$ for some $p > 1$.

Theorem (Iyer-Kiselev-Xu, '13): Let $\theta_0 \in L^\infty(\mathbb{T}^2)$, $\bar{\theta} = 0$, and let u be a regular, time-dependent flow. For any $p \in (1,\infty)$, $\lambda \in (0,1)$, $\exists$ $r_0 = r_0(\theta_0, \lambda)$, $\varepsilon_0 = \varepsilon_0(\lambda)$, $c = c(p)$ such that

$$\boxed{\mathcal{E}_f(\theta)(t) \geq \varepsilon_0 r_0^2 \|\theta_0\|_{L^\infty} \exp\left(\frac{-c}{m(A_\lambda)^{1/2}} \int_0^t \|\nabla u(s)\|_{L^p}\, ds\right)},$$

where $A_\lambda := \{x \in \mathbb{T}^2 \,/\, \theta_0(x) > \lambda \|\theta_0\|_{L^\infty}\}$ super-level set of θ_0.

Note constants depend on the size of level sets of θ_0, not just the L^∞ norm.

Remark : Independent proof by C. Seis ('13) using optimal transport for binary functions $\Rightarrow$ exponential lower bound on the Monge-Kantorovich distance $\mathcal{D}(\theta)$ (Brenier-Otto-Seis '11), plus interpolation inequality $C\,\|\theta\|_{TV}^{-1} \leq \mathcal{D}(\theta) \lesssim \mathcal{E}\ell(\theta)$, TV total variation.

Sketch of proof of theorem : ① Relax notion of ε-mixed set to ε-semi mixed set $\Rightarrow$ $\dfrac{m(A \cap B(x,\varepsilon))}{m(B(x,\varepsilon))} < 1-K$ for some $0<K<1/2$.

② $\exists\ C_0 = C_0(\lambda, K)$ such that $\|\theta_0\|^2_{H^{-1}} \leqslant \dfrac{\varepsilon^2}{C_0} \Rightarrow A_\lambda(\theta_0)$ is ε-semi mixed

③ If $\Phi_t(A_\lambda(\theta))$ is ε-semi mixed, then

$$\int_0^t \|\nabla u(t)\|_{L^p}\,dt \gtrsim \frac{m(A_\lambda)^{1/p}}{a} \log\left(\frac{2\varepsilon}{r_0}\right) \qquad (1<p<\infty)$$

④ Argue by contradiction. $\blacksquare$

Both proofs do not use energy estimates $\Rightarrow$ geometric measure theory arguments.

Exponentially mixing flows

Many classical examples of exponentially mixing maps (e.g. cat map, baker's map). Some examples of flows in dimension $d > 2$ on non-flat manifolds.

Here, we insist on flows with velocity of prescribed regularity.

Present geometric construction (Alberti – Crippa – M. '14, '19) that yields exponentially mixing flows with velocity $u \in W^{1,p}$, $1 \leq p \leq \infty$, for certain binary initial data.

This construction has applications to other problems: loss of regularity, anomalous dissipation.

As example of exponential mixers superceded by recent developments:

1. the flow generated by a time periodic, Lipschitz flow, alternating between independent piecewise linear shear flows is a (universal) exponential mixer (Elgindi - Liss - Mattingly '23, Myers Hill - Sturman - Wilson '21).

② Proof of ① relies on a perturbation argument and the fact that the time 1 image of alternating piecewise linear shear flows is a piecewise toral automorphism (under certain conditions) like the cat map.

③ The theory of random dynamical systems allows to construct exponentially mixing flows that are regular in space, but rough in time:

(a) Solutions of the 2D Navier-Stokes equations with stochastic forcing (white in time, colored in space) (Bedrossian-Blumenthal-Punshon Smith, '21)

(b) Pierrehumbert flow: alternating sine shear flows with random phase. (Blumenthal-Coti Zelati-Gvalani '22). Can also take fixed shears, but random intervals of time where they act (Cooperman '22).

④ All the examples in ①, ②, ③ are universal mixers (mix all initial conditions in a dense subset).

Self-similar and Quasi-self-similar exponential mixers

Describe a geometric approach to constructing flows that mix optimally binary functions (this last condition can be relaxed somewhat).

θ_0 will be of the form $\theta_0(x) = \begin{cases} -1, & x \in A^c \\ 1, & x \in A \end{cases}$, with $m(A) = \frac{1}{2} m(\mathbb{T}^2)$.

Prescribe the evolution of the set A. Show there exists a velocity field u that realize the given evolution.

Present two examples

(i) Sobolev example: velocity $u \in L^\infty([0,\infty); W^{1,p}(\mathbb{T}^2))$, $\forall\, 1 \le p < \infty$.
the evolution of set A contains a topological change (pinching singularity) and it is self-similar.

(ii) Lipschitz example: velocity $u \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{T}^2))$, the evolution of the set cannot have any topological change and it is quasi self-similar $\Rightarrow$ follows steps in the construction of the Peano curve (a space filling curve).

Related constructions: 1 Different analytic constuction of exponential mixers for functions that are not (close to) binary, using cellular flows as building blocks, $u \in L^\infty([0,\infty); W^{1,p})$, $p \approx 2$ (Yao–Zlatos, '17).

2 the construction in 1 was later generalized to an almost universal mixer, using the fact baker's map is the time 1 image of two shear flows, $u \in L^\infty([0,\infty); W^{s,p})$, $s \approx 1$, $p \approx 2$.

time-dependent paths and curves

View time as a parameter along families of curves in $\mathbb{T}^2$.

Notation: 1 paths: $\gamma : J \to \mathbb{T}^2$ (or $\mathbb{R}^2$), J interval in $\mathbb{R}$.
$s \mapsto \gamma(s)$

denote $\gamma(J)$ a curve. γ assumed at least of class C^1, ideally of class C^s, $s \geq 2$.

② time-dependent paths: $\gamma : J \times I \longrightarrow \mathbb{T}^2$ (or $\mathbb{R}^2$), with I, J intervals in $\mathbb{R}$.
$s, t \longrightarrow \gamma(s,t)$

Denote: $\frac{\partial \gamma}{\partial s} = \dot{\gamma}$, $\frac{\partial \gamma}{\partial t} = \gamma_t$ or $\partial_t \gamma$.

③ Adapted frame: $(\tau(s), \eta(s))$ for path ($(\tau(s,t), \eta(s,t))$ for time dependent paths, where τ is the tangent vector, η is the normal vector. With abuse of notation, write $\tau(s)$ for $\tau(\gamma(s))$, $\tau(s,t)$ for $\tau(\gamma(s,t))$ and similarly for η.

Orient all curves positively and choose $\eta(s) = -\tau(s)^{\perp} = -\frac{\dot{\gamma}(s)}{|\dot{\gamma}(s)|}$

the normal velocity v_n for a time-dependent path $\gamma(s,t)$ given by:

$$\boxed{v_n = \partial_t \gamma \cdot \eta}$$

④ time-dependent domains: $E : I \longrightarrow \mathbb{T}^2$ ($\mathbb{R}^2$), $E(t)$ class C^k, $k \geq 1$.
$t \longmapsto E(t)$

Define normal velocity v_n as outer velocity of $\partial E(t)$.

5 Compatible vector fields : u compatible with E if $u \cdot \eta = v_n$.

If u is regular and compatible and Φ is the flow of u:

$$\boxed{\Gamma(t) = \Phi(t, \Gamma(t_0)) \qquad E(t) = \Phi(t, E(t_0))}, \quad t, t_0 \in I,$$

where $\Gamma(t)$ is any connected component of $\partial E(t)$ (a Jordan curve)

$\Rightarrow$ $\theta(x,t) = \chi_{E(t)}(x)$ is a distributional solution of (T) with advecting velocity u and initial data $\theta_0 = \chi_{E(0)}$.

the construction of the exponential mixers based on the following lemma.

Smooth Evolution Lemma : Let E be a smooth time-dependent domain such that the measure of each connected components of $E(t)$ is conserved. then there exists a smooth, divergence free vector field u that is compatible with E.

Sketch of proof : use stream function ψ of u, $u = \nabla^{\perp}\psi \Rightarrow$ we can localize u by cutting off ψ, maintaining the divergence-free condition.

So it is enough to define ψ in a tubular neighborhood of $\partial E(t)$. Foliate this neighborhood with smooth curves $\Gamma_\alpha(s,t)$, $0 \le \alpha \le 1$. On each Γ_α, define $\psi(s,t) = \psi(x(s,t))$ as solution of the family of ODEs in s:

$$\boxed{\partial_\tau \psi(s,t) = v_n(s,t)}$$

where $\tau(s,t)$ is the tangent vector to Γ_α.

ψ is well defined as a function of $x \in \mathbb{T}^2$ if ψ periodic in s, which follows from following Lemma. $\blacksquare$

Lemma : Let Γ be a C^k (closed) curve and v a C^k function such that $\int_\Gamma v \, d\sigma = 0$. Given $\bar{r} > 0$, there exists u, autonomous, such that : a) $\underline{u \cdot n = v}$; b) $\underline{\operatorname{supp} u \subset \{x \,/\, \operatorname{dist}(x,\Gamma) < \bar{r}\}}$.

Proof of Lemma: let $\gamma : I \to \mathbb{R}^2$ be a parametrization of Γ.

choose: i) $\gamma(s_0) = x_0 \in \Gamma$; ii) $g : \mathbb{R} \to \mathbb{R}$ smooth, $g(0) = 1$, $\operatorname{Supp} g \subset [-\frac{1}{2}, \frac{1}{2}]$; iii) r with $0 < r < \bar{r}$.

set $B(\Gamma, r) := \{ x \,/\, \operatorname{dist}(x, \Gamma) < r \} \Rightarrow \exists\ C^k$ diffeomorphism $\Psi : I \times (-r, r) \to B(\Gamma, r)$. with $x = \Psi(s, y) \in B(\Gamma, r)$, define

$$\psi(x) = \psi(s, y) := g\left(\frac{y}{r}\right) \int_{s_0}^{s} v(\gamma(s'))\, \dot{\gamma}(s')\, ds'$$

$\Rightarrow \partial_\tau \psi = v$, $\psi \in C^{k-1}$, $\operatorname{Supp} \psi \subset B(\Gamma, \frac{r}{2}) \subset B(\Gamma, r)$.

then, extend ψ by zero to $B(\Gamma, \bar{r})$ and let $u = \nabla^\perp \psi$. $\blacksquare$

Remarks: ① Lemma extends to time-dependent curves if v is compatible, so that u is compatible.

② By the divergence theorem, condition $\int_\Gamma v \, d\sigma = 0$ is necessary to have u divergence free.

③ Choosing $v = \sigma n$ give existence of u compatible with Γ.

Homothetic curves : $\Gamma(t) = \lambda(t)\bar{\Gamma} = \{\lambda(t)x \mid x \in \bar{\Gamma}\}$ with $\lambda : I \to (0,+\infty)$, $\bar{\Gamma}$ given support curve. set $\bar{v} = x \cdot \bar{\eta}$, $\bar{\eta}$ normal to $\bar{\Gamma}$. then : $\eta(x,t) = \eta\left[\frac{x}{\lambda(t)}\right]$, $v_n(x,t) = \lambda'(t)\,\bar{v}\left(\frac{x}{\lambda(t)}\right)$.

Also, if $\bar{u}$ is compatible with $\bar{\Gamma}$ ($\bar{u}\cdot\bar{\eta} = \bar{v}$ on $\bar{\Gamma}$), $u(x,t) = \lambda'(t)\,\bar{u}\left(\frac{x}{\lambda(t)}\right)$ compatibile with Γ.

1st example · Pinching Singularity

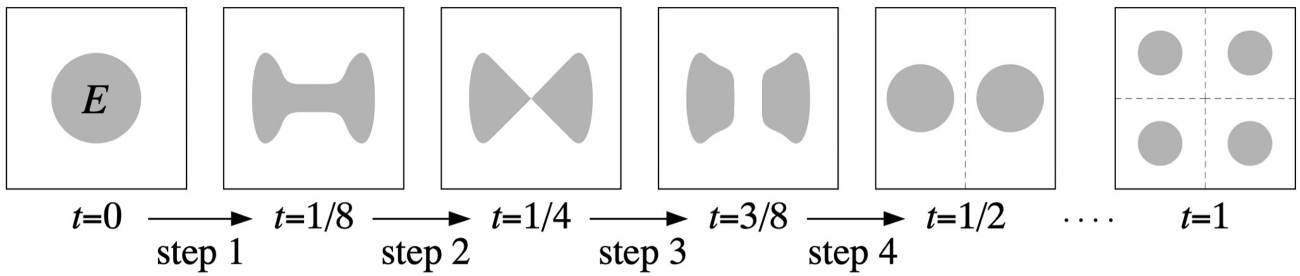
the mixing will be self similar. Only need to construct the first step. the iteration done by rescaling.

construct :

(a) $u^0 \in L^\infty([0,T]; W^{1,p}(\mathbb{T}^2))$, $1 \le p < \infty$, $T > 0$ (in fact, $u^0 \in L^\infty([0,T], W^{s,p})$ $s<1$, $1 \le p \le \infty$ or $s \ge 1$, $1 \le p \le \frac{2}{s-1}$ $\Rightarrow u_0(t) \notin Lip(\mathbb{T}^2)$).

(b) $\theta^0(t) = \chi_{E(t)} - \frac{\pi}{16}$, where E is a time-dependent set

Such that $m(E(t)) = \frac{\pi}{16}$, $E(0) = D(0, \frac{1}{4})$ disk, $E(1)$ is given by 4 copies of initial disk at scale $1/2$.



u^0, θ^0 smooth except at $t = \frac{k}{8}$, $k = 1, \dots, 7$.

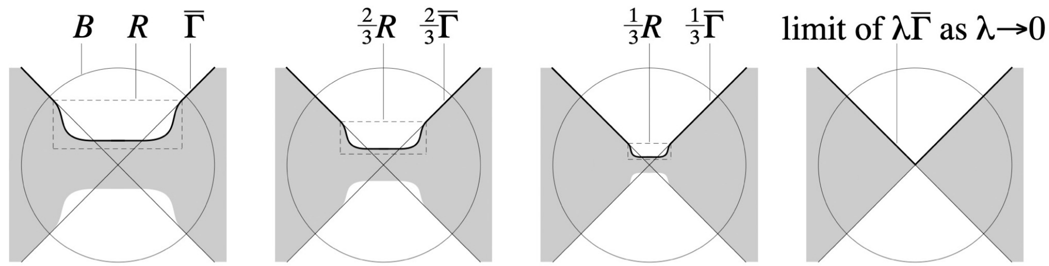
θ^0 continuous and transported by u^0 on intervals $\left(\frac{k}{8}, \frac{k+1}{8}\right)$.

$\Rightarrow$ θ^0 weak solution of $\partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = 0$.

Step 1 : construction of $E(t)$, $u^0(t)$, $0 \leq t \leq \frac{1}{8}$.

- Define $E(\frac{1}{8})$ by reflecting across vertical mid line (dotted)

Since $E(0)$, $E(\frac{1}{8})$ smooth, simply connected, same area, There exists a smooth map deforming $E(0)$ into $E(\frac{1}{8})$, preserving area $\Rightarrow$ $u^\circ(t)$ exists on $[0, 1/8]$ by Smooth Evolution Lemma, with support in a square $Q \subset \mathbb{T}^2$.



Step 2: construction of $E(t)$, $u^\circ(t)$, $\frac{1}{8} < t < 1/4$

- Let $\overline{\Gamma}$ be one of the two mirror-symmetric components of $\partial E(\frac{1}{8}) \cap R$, as in the figure. In $B \setminus R$, $\overline{\Gamma} = \{|x_1| = x_2\}$ and otherwise $\overline{\Gamma}$ smooth.

- Define homothety of $\overline{\Gamma}$ with factor $\lambda(t) : [\frac{1}{8}, \frac{1}{4}) \to (0, 1]$ decreasing $\lambda(\frac{1}{8}) = 1$, $\lambda \xrightarrow[t \to \frac{1}{4}^-]{} 0$.

- Enough to construct $\partial E(t)$ (Jordan curve) so that:

(a) $E(t) = E(\frac{1}{8})$ on $Q \setminus B$;

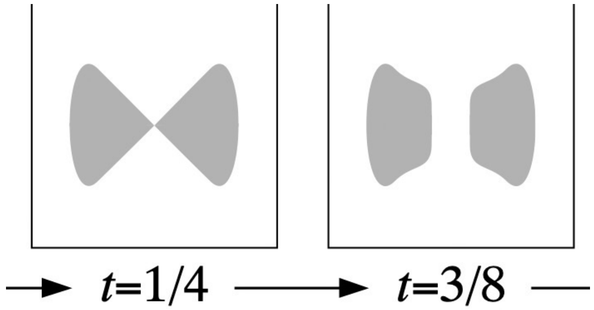
(b) $\partial E(t) \cap B$ has two mirror-symmetric components, each agreeing (up to rigid motions) with $\lambda(t)\bar{\Gamma}$ in B

- By smooth Evolution Lemma, $\exists\ u^0$ in a neighborhood of $\lambda(t)\bar{\Gamma}$ for $t \in (\frac{1}{8}, \frac{1}{4})$. Extend it by reflection in R, and by zero to Q, since $E(t) = E(\frac{1}{8})$ on R^c.

- u^0 is of the form $u^0(x,t) = \lambda'(t)\,\tilde{u}\left(\frac{x}{\lambda(t)}\right) \Rightarrow$ choose $\lambda(t)$ so that u^0 has needed Sobolev regularity $\left(\lambda(t) = e^{2 - \frac{1}{1-4t}}\right)$.

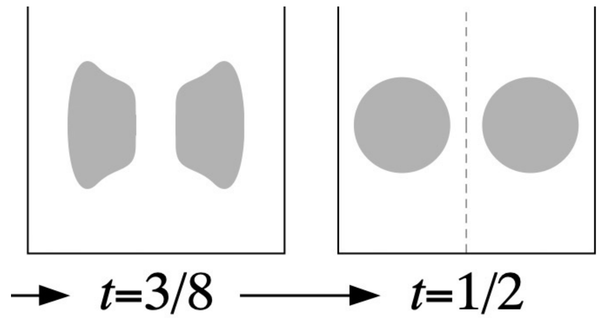
Step 3: Construction of $E(t)$, $u^0(t)$ on $\frac{1}{4} \leq t \leq \frac{3}{8}$.

- Proceed similarly to Step 2 with $E(\frac{1}{4})$, $E(\frac{3}{8})$ as given in the figure, using homothety.



Step 4: Construction of $E(t)$, $u^0(t)$ on $\frac{3}{8} \leq t \leq 1/2$

- Proceed similarly to step 2 with $E(\frac{3}{8})$, $E(\frac{1}{2})$ as given in the figure.



- the two disk are exact copies of $D(0, \frac{1}{4})$ with radius $\frac{1}{8}$.

Step 5: construction of $E(t)$, $u^0(t)$ on $\frac{1}{2} \leq t \leq 1$

- Repeat steps 1-4 on each of the two disks to create 4 identical disks of radius $\frac{1}{16}$.

Iteration and construction of u, θ

- Define $u^n(x,t) = 2^{-n} u^0\left(t-n, \frac{x}{2^n}\right)$, $\theta^n(x,t) = \theta^0\left(t-n, \frac{x}{2^n}\right)$, $n \in \mathbb{N}$
- Let $u(x,t) = u^n(x,t)$, $\theta(x,t) = \theta^n(x,t)$ on $[n, n+1) \times \mathbb{T}^2$

$\Rightarrow$ θ weak solution of (T) with velocity u on $(0,+\infty)\times\mathbb{T}^2$ with $\theta(0)=\theta^0(0)$.

- By scaling (note we do not rescale the domain):

$$\|\theta(n)\|_{H^{-1}} = \|\theta^n(n)\|_{H^{-1}} = 2^{-n}\|\theta^0(0)\|_{H^{-1}} \xrightarrow[n\to\infty]{} 0$$

Remarks: ① this example show pathologies that regular Lagrangian flows, arbitrarily close to Lipschitz, can have:

(a) Flow can compress a segment to a point (expand a point to a segment) in finite time.

(b) trajectories of u starting at any point of this segment are non unique.

② Construction is localited near $\partial E(t)$, in the cube $Q\subset\mathbb{T}^2$ $\Rightarrow$ it can be adapted to the case of $\mathbb{R}^2$ (with u,θ still compactly supported) or a bounded domain with compatible boundary conditions.

2nd Example : Peano Snake

Construction follows similar ideas as for the pinching singularity
$\Rightarrow$ give the time evolution of a set $E(t)$ and use the smooth Evolution Lemma to construct u_0.

Here : • Construction is quasi self-similar : $E(1)$ is not an exact replica of $E(0)$ at smaller scale. It is a suitable combination of rescaled copies from a finite family of initial patterns.

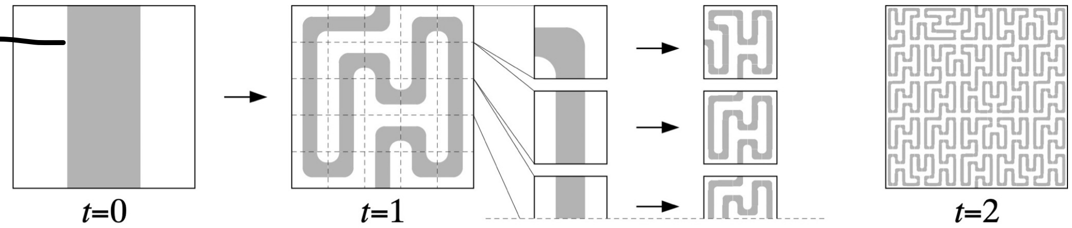
• the initial condition is a strip centered around the median segment in $\mathbb{C}^2$.

• the time evolution follows the iterative construction of the Peano Curve, a space filling curve.

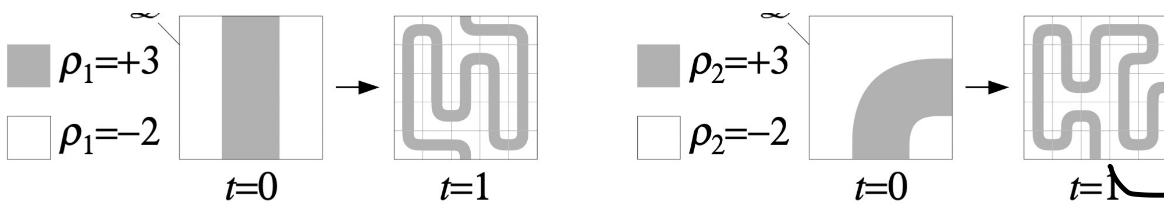
• Although u can be made smooth (derivative jumps because of periodicity) control only Lipschitz norm uniformly in time.

Evolution of the set $E(t)$ in Example 2:

jump can be a smooth transition



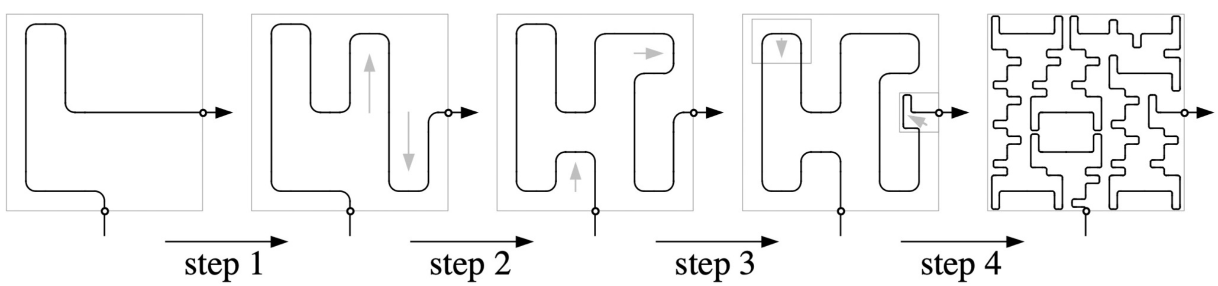
Family of basic moves for time stepping:



∇u could be disconti-nuous only at entry/exit points.

Set $E(t)$ constructed using homothety.

time evolution of support curve of $E(t)$:



IV. LOSS OF REGULARITY IN LINEAR TRANSPORT EQUATIONS

Optimal mixers useful to investigate the ill-posednessness of linear transport equations with rough (but not too rough) velocities.

- we have already shown with Example 1 pointwise discontinuity of the flow map
- Investigate discontinuity in Sobolev spaces. $\Rightarrow$ byproduct of loss of regularity for solutions of (T).

Remark - Nonuniqueness of weak solutions

Recall the "slide-and-dice" example of finite-time perfect mixing.

By linearity ($\theta = 0$ always solution) and time reversibility,

finite-time perfect mixing $\Rightarrow$ nonuniqueness for solutions to (T)

In the slice-and-dice example, $u(t) \in BV(\mathbb{T}^2)$ up to $t = T_{mix}$, and the total variation of u is proportional to the length of the interfaces being created in the tracer field.

$\Rightarrow$ $\|u(t)\|_{TV}$ doubles on each successive intervals of time of length 2^{-n}.

Since $T_{mix} = \sum \frac{1}{2^n}$, $\|u(t)\|_{TV} \approx \frac{1}{T_{mix}-t}$, $0 < t < T_{mix}$

$\Rightarrow$ $u \in L^{1,\infty}([0, T_{mix}); BV)$

By Ambrosio's result ($u \in L^1((0,T); BV) \Rightarrow$ uniqueness), the slide-and-dice example is optimal.

Loss of regularity

Since mixing by stirring alone is obtained by creating small scales in the tracer field (= large derivatives), one expects a connection with growth of Sobolev norms $\Rightarrow$ encoded in the interpolation inequality:

$$\boxed{\|\theta(t)\|^2_{L^2} \leq \|\theta(t)\|_{H^s} \|\theta(t)\|_{H^{-s}}} \quad \forall s \geq 0$$

$\|\theta(t)\|_{H^{-s}} \to 0 \Rightarrow \|\theta(t)\|_{H^s} \to \infty$, since $\|\theta(t)\|_{L^2}$ constant.

Modifying the "Peano Snake" example gives the following result.

Theorem (Crippa - Alberti - M. '19): Let $d \geq 2$. There exist $\theta_0 \in C_c^\infty(\mathbb{R}^d)$ and $u \in L^\infty([0,\infty) \times \mathbb{R}^d)$ such that:

(i) $u \in L^\infty([0,\infty); W^{1,p}(\mathbb{R}^d))$, $\forall\, 1 \leq p < \infty$;

(ii) if θ is the unique weak and Lagrangian solution of (T) with velocity u and $\theta(0) = \theta_0$, then $\theta \in L^\infty([0,+\infty) \times \mathbb{R}^d)$

iii) $\|\theta(t)\|_{H^s(\mathbb{R}^d)} = \infty \quad \forall\, t > 0, \ \forall\, s > 0$.

In addition, u and θ are supported on a cube in $\mathbb{R}^d$ and smooth outside a point $x_0 \in \mathbb{R}^d$ in space.

Remarks. ① The theorem provides an example of total, instantaneous loss of Sobolev regularity (including fractional) for weak solution to linear transport equations, and 1st example of its kind.

② θ is the unique renormalized (hence Lagrangian) solution $\Rightarrow$ theorem implies discontinuity of the flow map in $W^{1,p}$, $1 \leq p < \infty$.

Independently, Jabin ('15) showed directly discontinuity of the flow map in $W^{1,p}$ by using a random flow. (see also De Nitti - Bianchini '20).

③ By contrast, if u is Lipschitz, then the flow map is also Lipschitz (though Lipschitz constant can grow exponentially in time) and regularity of θ up to Lipschitz is propagated.

④ Some regularity of θ does get propagated by u with $W^{1,p}$ regularity
=> essentially only the logarithm of derivatives ($\approx$ Fourier multiplier $\log|\xi|$, Léger '18) is propagated and our example implies that this result is sharp (Brué - Nguyen '19).

⑤ Loss of regularity is in fact a generic phenomenon, in the sense of Baire's category theorem (Ghisi - Gobbino '20, Bianchini - Zizza '22)

⑥ Some connections with norm inflation phenomena for PDEs, but here it is a linear phenomenon.

Main idea of proof: use mixing to grow Sobolev norms exponentially, then rescale to turn growth into instantaneous blow up.

We will need a technical lemma to treat fractional regularity $\Rightarrow$ although the H^s norm, $0<s<1$, is not local, it almost decouples for superpositions of functions with well separated supports.

To prove Lemma we use Gagliardo seminorms in $H^s(\mathbb{R}^d)$, $0<s<1$:

$$\boxed{\|f\|^2_{\dot{H}^s} \approx \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,dx\,dy}$$

Main reason why we use L^2-based Sobolev norms for Θ.

<u>Lemma</u>: Let $0<s<1$, $K_i \subset\subset \Omega_i \subset \mathbb{R}^d$, Ω_i open, K_i compact, $\Omega_1 \cap \Omega_2 = \phi$, $\operatorname{dist}(K_i, \Omega_i^c) =: \lambda_i > 0$, $i=1\ldots N$, $N\in\mathbb{N}$. If $f_i \in H^s(\mathbb{R}^d)$, $\operatorname{supp} f_i \subset K_i$, $i=1\ldots N$, then

$$\boxed{\Big\|\sum_{i=1}^N f_i\Big\|^2_{H^s} \geq \sum_{i=1}^N \|f_i\|^2_{H^s} - \frac{C(d)}{s}\sum_{i=1}^N \frac{1}{\lambda_i^{2s}}\|f_i\|^2_{L^2}}$$

Formula extends to <u>series</u> if RHS is positive (our case).

Remark: the construction of 2D exponential mixers can be lifted to any $d \geq 2$ in a straightforward fashion. Given $\eta \in C_c^\infty(\mathbb{R}^{d-2})$, let:

$$\boxed{\begin{aligned} \bar{u}(x_1 \dots x_d) &= \eta(x_3, \dots, x_d)\, u(x_1, x_2) \\ \bar{\theta}(x_1 \dots x_d) &= \eta(x_3, \dots, x_d)\, \theta(x_1, x_2) \end{aligned}}$$

Sketch of proof of theorem: We construct u and θ as sums $u = \sum_n u^{(n)}$, $\theta = \sum_n \theta^{(n)}$, where $u^{(n)}$, $\theta^{(n)}$ are obtained by rescaling $u^{(0)}$, $\theta^{(0)}$.

Step 1: construction of basic elements $u^{(0)}$, $\theta^{(0)}$

The construction of the Lipschitz exponential mixer ("Peano snake") can be modified to make velocity and the scalar smooth. Then lift them to $u^{(0)}$, $\theta^{(0)}$ in $\mathbb{R}^d$, supported on the unit cube $Q_0 \subset \mathbb{R}^d$, $u^{(0)}$ divergence free.

From the construction the following norm bounds hold:

(a) $u^{(0)}, \theta^{(0)} \in L^\infty([0,+\infty)\times\mathbb{R}^d)$, $\fint_Q \theta^{(0)}\,dx = 0$;

(b) $u^{(0)} \in L^\infty([0,+\infty); W^{1,p}(\mathbb{R}^d))$, $\forall\ 1\le p<\infty$, and $\forall\ r\geq 0$, $\exists\ b=b(p)>0$, $B_r = B_r(p,r)>0$ such that

$$\boxed{\|u^{(0)}(t)\|_{\dot{W}^{r,p}} \le B_r\, e^{b(r-1)t}, \quad t>0;} \qquad (*)$$

(c) $\forall\ 0\le s<2$, $\exists\ c>0$, $\hat{C}_s = \hat{C}_s(s)>0$ such that

$$\boxed{\|\theta^{(0)}(t)\|_{H^{-s}(\mathbb{R}^d)} \le \hat{C}_s\, e^{-cst}, \quad t>0;} \qquad (**)$$

the $L^2 - H^s$ interpolation inequality then implies

$$\boxed{\|\theta^{(0)}(t)\|_{H^s(\mathbb{R}^d)} \geq C\, e^{cst}}, \quad t>0, \qquad (***)$$

for some constant $C = C(p, d, s, \theta^{(0)}(0))$

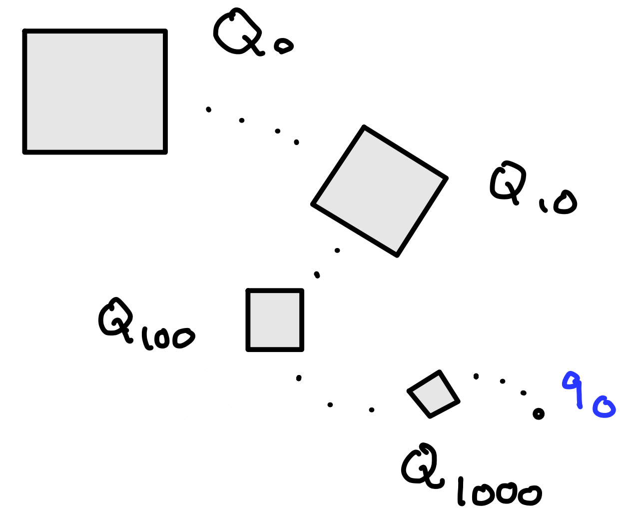
Step 2 : construction of $\theta^{(n)}, u^{(n)}$

- Let $\{\lambda_n\}$ be a sequence of positive numbers, $\lambda_n \to 0$ as $n\to\infty$,

to be chosen later. Let $Q_n = 3\lambda_n Q_0$ (up to rigid motion).
Choose centers of cubes Q_n so that they are pairwise disjoint
$Q_n \cap Q_m = \emptyset$, if $n \neq m$ and such that in the sense of
convergence of sets $Q_n \xrightarrow[n\to\infty]{} \{q_0\}$, a point in $\mathbb{R}^d$.

want u, θ to have compact support
in $\mathbb{R}^d \Rightarrow$

(A) $m\left(\bigcup_n Q_n\right) < \infty$ if $\sum_n \lambda_n < +\infty$.



- Up to translations and rotations, set:

$$u^{(n)}(x,t) = \frac{\lambda_n}{\tau_n}\, u^{(0)}\left(\frac{t}{\tau_n}, \frac{x}{\lambda_n}\right)$$

$$\theta^{(n)}(x,t) = \gamma_n\, \theta^0\left(\frac{t}{\tau_n}, \frac{x}{\lambda_n}\right)$$

$\Rightarrow$ $\left.\begin{array}{l}\operatorname{Supp} u^{(n)} \\ \operatorname{Supp} \theta^{(n)}\end{array}\right. \subset Q_n$

for sequences $\{\tau_n\}, \{\gamma_n\}$ of positive numbers, to be chosen later.

- Meaning of parameters: $\begin{cases} \lambda_n & \text{space scaling} \\ \tau_n & \text{time scaling} \\ \gamma_n & \text{amplitude scaling} \end{cases}$

Step 3 : construction of u, θ

- Let $u = \sum u^{(n)}$, $\theta = \sum_n \theta^{(n)}$ $\Rightarrow$ u, θ well defined at least a.e. (Q_n have pairwise disjoint support).

$\theta^{(0)}$ weak solution of (T) with velocity $u^{(0)}$ $\Rightarrow$ $\theta^{(n)}$ weak solution of (T) with velocity $u^{(n)}$ $\Rightarrow$ θ weak solution of (T) with velocity u.

Step 4 : check norm bounds

- From behavior of Lebesgue and Sobolev under rescaling in $\mathbb{R}^d$:

(B) $u \in L^\infty([0,+\infty); \dot{W}^{1,p})$ if $\displaystyle\sum_n \frac{\lambda_n^{1-r+\frac{d}{p}}}{\tau_n} e^{-\frac{(r-1)bt}{\tau_n}} < \infty$

($\tilde{B}$) $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ if $0 \leq \dfrac{\lambda_n}{\tau_n} \leq C$, C independ. of n

using estimate (*).

- Using also Lemma on localization of H^s norms:

Ⓒ $\theta_0 = \theta(0) \in H^\sigma(\mathbb{R}^d)$ $\forall \sigma$ if $\sum_n \gamma_n \lambda_n^{\frac{d}{2}-\sigma} < +\infty$

$\tilde{C}$ $\theta \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ if $\{\gamma_n\}$ bounded.

- Using also (***)

Ⓓ $\theta(t) \notin H^s(\mathbb{R}^d)$, $s>0$, $t>0$ if $\sum_n \gamma_n^2 \lambda_n^{d-2s} e^{\frac{2sct}{\tau_n}} = \infty$

Step 5: Choice of $\lambda_n, \tau_n, \gamma_n$

- Choose $\tau_n = \frac{1}{n^3}$, $\lambda_n = e^{-n}$ $\Rightarrow$ Ⓐ, Ⓑ, $\tilde{B}$ hold with $r=1$ for all $1 \leq p < \infty$.

- Choose $\gamma_n = e^{-n^2}$ $\Rightarrow$ Ⓒ, $\tilde{C}$ hold with $\sigma > 0$.

- Verify that Ⓓ holds with these choices of parameters.

Condition (D) becomes : $\sum_n e^{-2n^2} e^{-(d-2s)n} e^{2cst\,n^2} = +\infty$,

since $cst > 0 \Rightarrow$ (D) holds.

$\boxtimes$

• Two natural questions arise :

① Does loss of regularity holds for all Sobolev spaces that does not embed in the Lipschitz space, i.e. for $u \in W^{r,p}(\mathbb{R}^d)$, $1<r<\frac{d}{p}+1$, $1\leq p<\infty$?

② Does there exists a universal construction for u that makes (most) initial conditions θ_0 blow-up?

We cannot take $r>1$ in the present construction, as scaling is unfavorable in this regime $\Rightarrow$ norms of u grow for $t>0$.

We give partial answer to ① and ② without appealing to mixing flows.

Key idea: blow-up of positive norms is a local phenomenon, growth can be achieved with simple flows that are not mixing $\Rightarrow$ allow for explicit computation of the growth of norm in time and allow for more flexible rescaling.

Loss of regularity revisited

theorem (Crippa - Elgindi - Iyer - M. '22): Let $\theta_0 \in H^1_{loc}(\mathbb{R}^d)$, $d \geq 2$, non-constant. There exists a compactly supported, div. free, vector field $u \in L^\infty([0,\infty); W^{r,p}(\mathbb{R}^d))$, $1 \leq p < \infty$, $r < \frac{d}{p} + 1$, such that the unique weak solution of (T) with velocity u and $\theta(0) = \theta_0$ satisfies: $\boxed{\theta(t) \notin H^1_{loc}(\mathbb{R}^d) \quad \forall\, t > 0}$.

Remarks: ① We do not know how to show that the fractional norms H^s, $0 < s < 1$, explode, since the growth of H^1 norm is an explicit energy estimate.

(2) the proof is still based on rescaling of a basic element, but the location where the rescaling occurs can no longer be arbitrary, but it is based on where the H^1 norm of θ_e is large

(3) the basic element is constructed from following observation:
the H^1 norm of a non-constant function φ on $\mathbb{T}^d$ increases by a fixed amount under the action of shear flows parallel to axes at time 1.

(4) Recently, instantaneous loss of some regularity was established for the 2D Euler equations in vorticity form (Cordoba-Martinez Zoroa-Ożański, '22) and even for 2D surface quasi-geostrophic (SQG) equation for fractional dissipation (Cordoba-Martinez Zoroa, '23) by a related norm inflation + rescaling + gluing procedure for some initial conditions.
2D Euler and (inviscid) SQG are both active scalar equations.

The observation in (3) follows from an explicit calculation.

Notation: Set $f_i(z) = A \sin\left(2\pi z + (i-1)\frac{\pi}{2}\right)$, $i = 1, 2$, $A > 0$. Let e_k be the elements of standard basis $e_k = (0, \dots, 1, \dots 0)$, $k = 1, \dots, d$.
↪ kth entry

<u>Lemma</u>: Let $\Omega_0 \subset \mathbb{T}^d$, $d \geq 2$, be a given C^1 subdomain. For any non-constant function $\psi \in H^1(\mathbb{T}^d)$, $\exists$ a vector field u (which depends on $\psi \chi_{\Omega_0}$) such that:

i) u is a <u>shear</u> flow $\boxed{u(x) = \pm f_i(x_j) e_{j'}}$ with $i = 1$ or 2

for some $j = 1 \dots d$ and $j' = \begin{cases} j+1, & j < d \\ 1, & j = d. \end{cases}$

ii) If ϕ is the weak solution of $\partial_t \phi + u \cdot \nabla \phi = 0$, $\phi(0) = \psi$ on $\mathbb{T}^d$, then for $T > 0$:

$$\boxed{\|\nabla \phi(\cdot, T)\|_{L^2(\Omega_T)} \geq \left(1 + \frac{2\pi^2 A^2 T^2}{d}\right) \|\nabla \phi\|^2_{L^2(\Omega_0)}}$$

where Ω_T image of Ω_0 under flow of u at $t = T$.

<u>Proof of Lemma</u>: For $i, i' \in \{1, 2\}$, $j \in \{1, \dots, d\}$, set

$u_{i,i',j}(x) := (-1)^i f_{i'}(x_j) e_{j'}$, and let $\phi_{i,i',j}$ be ψ

transported by $u_{i,i',j}$: $\phi_{i,i',j}(x,t) = \psi(x - (-1)^i f_{i'}(x_j) e_{j'})$

Computing derivatives and summing over i, i', j:

$$\sum_{i,i',j} \| \nabla \phi_{i,i',j} \|^2_{L^2(\Omega_{T,i,i',j})} = (4d + 8\pi^2 A^2 T^2) \| \nabla \psi \|^2_{L^2(\Omega_0)}$$

Since there are $4d$ terms on the left, at least one must be $> \frac{1}{4d}$ of the right-hand side, which gives the result. $\boxed{/\!/}$

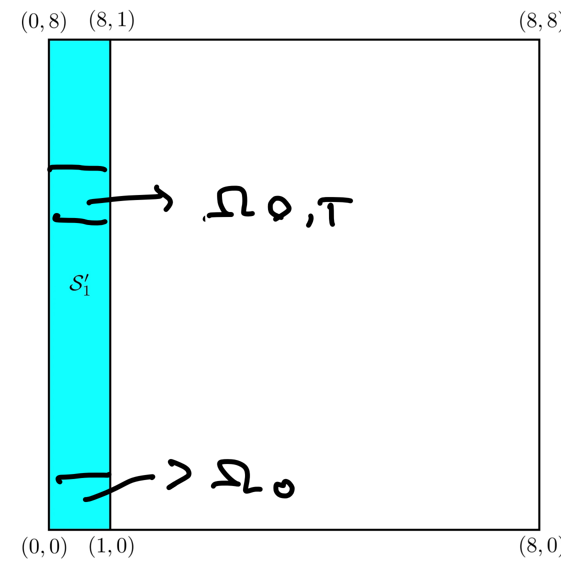
Step 1: exponential growth of H^1 norm

Using Lemma, given any $\theta_0 \in H^1_{loc}(\mathbb{R}^d)$, construct smooth (in x), compactly supported, div-free vector field such that the H^1-norm of $\theta(t)$ grows exponentially in t.

this flow and the weak solution it generates are initial elements of an iterative rescaling scheme.

to apply Lemma, we lift the flow with velocity u from the torus $\mathbb{T}^d$ to $\mathbb{R}^d$.

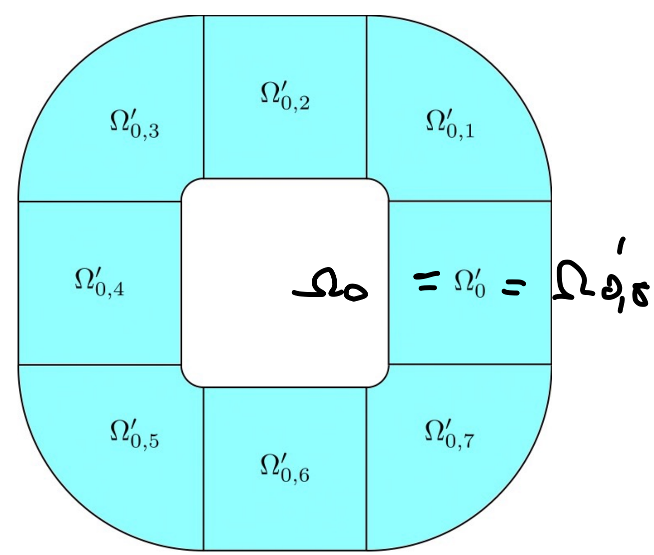
Describe lifting only for $d=2$. Identify $\mathbb{T}^2$ with $[0,8]^2$ and choose $\Omega_0 = [0,1]^2 \subset [0,8]^2$.



- By Lemma, a vertical or horizontal shear grows the H^1-norm of θ_0.

  Say it is a vertical shear.

- Then the image of $\Omega_0 = [0,1]^2$ under this shear lies in a vertical strip in $[0,8]^2$ (the strip S_1).



- Deform S_1 into the closed "track" er by periodicity, keeping Ω_0 fixed

$\Rightarrow$ the C^1 norm of the resulting flow is controlled

- By Lemma norm of $\theta(T)$ grows in at least one subdomain $\Omega_{0,j}'$, which up to a rotation can be identified with $\Omega_8' = \Omega_0' = \Omega_0 \Rightarrow$ growth in Sobolev norm.

<u>Proposition</u>: Let $\theta_0 \in H^1_{loc}(\mathbb{R}^d)$, $d \geq 2$, and fix $\alpha > 0$. Then, $\exists\ C = C(d,\alpha)$, <u>independent of θ_0</u>, and a vector field v, div-free, supported on $\overline{\Omega}_0 = [-3,4]^d$, $\forall t \geq 0$, such that:

$$\boxed{\sup_{t \geq 0} \|v(t)\|_{C^1(\mathbb{R}^d)} \leq C(d,\alpha).}\ , t \geq 0,$$

and the weak solution θ of (T) with velocity v, initial condition θ_0, satisfies:

$$(a)\ \boxed{\|\nabla\theta(n)\|_{L^2(\Omega_0)} \geq e^{\alpha n} \|\nabla\theta_0\|^2_{L^2(\Omega_0)}}\ , \forall n \in \mathbb{N},$$

with $\Omega_0 = [0,1]^d$, and

(b) $\boxed{\|\nabla\theta(t)\|_{L^2(L_0)} \geq e^{\alpha t - \beta} \|\nabla \theta_0\|_{L^2(\Omega_0)}}$, $\forall\, t \geq 0$,

for $\beta = \beta(\alpha, d)$ independent of θ_0.

Sketch of the proof:

- lift flow u from $\mathbb{T}^d$ to $\mathbb{R}^d \Rightarrow v$ resulting flow.
- composing flow with itself gives exponential growth of the H^1-norm of θ at $t = n$.
- Use that $\|v(t)\|_{C^1}$ is uniformly in t to get lower bound at intermediate times $t \in (n, n+1)$ up to a small loss.

$\square$

Step 2: Scaling and iteration

Use lifted velocity v and the associated weak solution θ.

Pick a sequence of cubes Q_n, center c_n, and sidelength λ_n

Pick $\tilde{Q}_n$ such that $\tilde{\tilde{Q}}_n := 7Q_n$ are pairwise disjoint and cluster at a point $y \in T^*$. the precise location is to be chosen later on.

- up to a rigid motion, we can repeat steps 1)-4) and construct v on Q_n. call $\boxed{v_n = v|_{Q_n}}$ $\Rightarrow$

  v_n grows norms of θ exponetially.

Define $u = \sum_n u_n$ and $\theta = \sum_n v_n$ $\Rightarrow$ θ weak solution of (T) with velocity $u = u_0$, initial condition θ_0

- Rescale v_n to achieve blow up: $u_n(x,t) = \frac{\lambda_n}{\tau_n} u\left(\frac{t}{v_n}, \frac{x}{\lambda_n}\right)$

- we have $\operatorname{Supp} u_n \subseteq Q_n$, u smooth in x outside of a point q_0 (where $\tilde{Q}_n$ concentrate as $n \to \infty$).

- Define $u = \sum_n u_n$ and let θ be the weak solution of (T) with velocity u, initial data θ_0.

- By construction:

$$(\diamond) \quad \boxed{\begin{aligned} &\|u_n(t)\|_{W^{r,p}(\mathbb{T}^d)} \lesssim \sum_{n=1}^{\infty} \frac{\lambda_n^{\gamma}}{c_n}, \qquad \gamma = 1 - r + \frac{d}{p} \\ &\|\nabla \theta_n(t,\cdot)\|_{L^2(\tilde{Q}_n)} \gtrsim \sum_{n=1}^{\infty} e^{\frac{\alpha t}{\tau_n}} M_n, \qquad M_n = \|\nabla \theta_0\|_{L^2(\tilde{Q}_n)} \end{aligned}}$$

Goal is to choose c_n, λ_n, τ_n so that the first inequality above is $< \infty$, the second $= \infty$.

Step 3: Covering Lemma, Choice of λ_n, c_n

Choose cubes Q_n based on where a rescaled local version of H^1 norm of θ_0 is large

Let $f(x) = |\nabla \theta_0(x)|^2 \Longrightarrow f \in L^1_{loc}(\mathbb{R}^d)$, $f \neq 0$.

Define

$$\boxed{A_r(x) := \frac{1}{|Q_r(x)|} \int_{Q_r(x)} f(y)\,dy}$$

Set $\tilde{D} = \{ x \in \mathbb{R}^d \;/\; \exists \lim_{r \to 0+} A(r) = f(x) \}$.

$\tilde{D}$ has full measure by Lebesgue differentiation theorem, and $\exists\, \bar{\delta} > 0$ (since $f \neq 0$) such that the following subset of $\tilde{D}$, $D := \{ x \in \tilde{D} \;/\; \lim A_r(x) \geq \bar{\delta} \} \cap B(0,R)$, $R > 0$ has positive measure $\Rightarrow$

for $x \in D$, $\exists\, r_x > 0$ such that $\int_{Q_r(x)} f(y)\,dy \geq \frac{\bar{\delta}}{2} r^d$, $\forall\, 0 < r < r_x$, where $Q_r(x)$ cube with center x, side r.

$\Rightarrow \exists\, c_n \in D$, $\lambda_n > 0$ such that $\begin{cases} 0 < \lambda_n < e^{-n} \\ M_n \geq c\, \lambda_n^{d/2} \end{cases}$, and

$Q_n = Q_{\lambda_n}(c_n)$ have property that $\tilde{Q}_n = \mathcal{Z}\, Q_n$ are pairwise disjoint.

Finally, since D bounded, $\{c_n\}$ has a cluster point ($\tilde{Q}_n$ accumulate to a point) and $\bigcup_n \tilde{Q}_n$ is bounded

Step 4: choice of τ_n

From estimate $(\lozenge)$, $\|\theta(t)\|_{\dot{H}^1_{loc}} = \infty$ if

(B.1) $\displaystyle\sum_{n=1}^{\infty} e^{t/\tau_n} \lambda_n^{d/2}$, $t > 0$; while $\|u(t)\|_{\dot{W}^{r,p}} \leq C$, if

(B.2) $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda_n^{\gamma}}{\tau_n} \leq C$, $t > 0$, $\forall\ \gamma = 1 - r + \frac{d}{p} > 0$.

Choose $\tau_n = \left(\log \frac{1}{\lambda_n}\right)^{-2}$ $\Rightarrow$ can verify (B.1) by a direct calculation. For (B.2), $\exists\ N = N(\gamma)$ such that

$$\left(\log \frac{1}{\lambda_n}\right)^2 \leq \left(\frac{1}{\lambda_n}\right)^{\gamma/2}, \quad \forall\, n \geq N(\gamma) \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{\gamma}}{\tau_n} \leq \sum_{n=1}^{N(\gamma)-1} \left(\log \frac{1}{\lambda_n}\right)^2 \lambda_n^{\gamma} + \sum_{n=N(\gamma)}^{\infty} e^{-\gamma n/2} < \infty$$

Open problems

① Can we modify construction to show loss of H^s norm $0<s<1$? Interpolation requires a lower bound on H^{-s} $\Rightarrow$ mixing.

② Can we construct a universal "exploder"? Idea is to replicate this construction on a sufficiently dense set in $\mathbb{R}^d$, but a challenge is that cubes $\widetilde{Q}_n$ are no longer disjoint.

IV: ENHANCED DISSIPATION

- Consider linear advection-diffusion equation:

$$\partial_t \theta + \vec{u}\cdot\nabla\theta - \nu\,\Delta^{\gamma}\theta = 0, \quad \theta(0)=\theta_0, \quad \gamma = 1,2 \qquad (\text{ADE})$$

with $\Omega = \mathbb{R}^d$, or Ω bounded with periodic or homog. Dirichlet/Neumann b.c., u tangent to $\partial\Omega$, $\theta_0 \in L^2(\Omega)$

- Denote by $\mathcal{S}(t,s)$, $0 \leq s \leq t$, the solution operator of (ADE). take θ_0 mean-free $\Rightarrow$ uniqueness.

- set $\vec{u} = A\vec{U}$, $A>0$ amplitude. time change $\Rightarrow A=1$
  $A \to \infty \iff \nu \to 0$

- Define dissipation time $\tau = \tau(u)$ of (flow of) u:

$$\tau := \operatorname{Inf}\left\{ t>0 \,;\, \|S(t+s,s)\,\theta(s)\|_{L^2} \leq \frac{1}{2}\|\theta(s)\|_{L^2},\ s \geq 0 \right\}$$

Fact: $0 < \tau < \infty$, $\tau(u) \leq \tau(0)$.

Enhanced Dissipation cont.

- Say that $\vec{u}$ is dissipation enhancing if

$$\tau(\vec{u}) \to 0 \quad \text{as} \quad \nu \to 0$$

- $\vec{u}$ dissipation enhancing $\Rightarrow$ dissipation timescale $o(\sqrt{\nu})$

$$\Rightarrow \quad \| S(t,0) \|_{op} \leq e^{-H(\nu)(t)}, \qquad H(\nu)/\sqrt{\nu} \xrightarrow[\nu \to 0]{} \infty .$$

Examples: ① steady flows satisfying a certain spectral condition $\Longrightarrow$ no H^γ eigenfunctions (relaxation enhancing flows, Constantin - Kiselev - Ryzhik - Zlatos)

② mixing flows $\Longrightarrow$ $\|\theta(t)\|_{H^{-1}} \lesssim h(t) \|\theta_0\|_{H^1}$

(Coti Zelati - Delgadino - Elgindi, Feng - Iyer, ...)

③ certain shear or cellular flows for prepared data (Iyer - Xu - Zlatos, Bedrossian - Coti Zelati, ...)

Resolvent estimates

- In the literature, enhanced dissipation proved by essentially 2 methods: hypocoercivity or resolvent estimates (but also probabilistic methods).

- Hypocoercivity more difficult to adapt to the case of hyper or fractional dissipation, and when Ω unbounded (lack of Poincaré's Inequality)

- Resolvent estimates may lead to more restrictive conditions on u.

- Exploit a Gearhart-Prüss type result for maximally accretive (m-accretive) operators on Hilbert spaces.

m-Accretivity and semigroup decay rates

- A densely defined linear operator $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ on a complex Hilbert space $(\mathcal{H}, \langle , \rangle)$ is called m-accretive if

  1. $\mathrm{Re} \langle Hf, f \rangle \geq 0 \quad \forall f \in \mathcal{D}(H)$ (accretivity)
  2. $\mathrm{Range}\,(H + \xi\, \mathrm{Id}) = \mathcal{H}$, for some $\xi > 0$ (maximality).

- If H is a closed, m-accretive operator on $\mathcal{H}$, then (Wei):

$$\| e^{-tH} \|_{op} \leq e^{\frac{\pi}{2} - t\,\Psi(H)}, \quad t \geq 0,$$

where $\|\cdot\|_{op}$ is the operator norm and

$$\Psi(H) := \mathrm{Inf} \left\{ \|(H - i\lambda) g\| \,/\, g \in \mathcal{D}(H),\ \lambda \in \mathbb{R},\ \|g\| = 1 \right\}$$

with $\|\cdot\|$ the Hilbert space norm in $\mathcal{H}$.

- goal : to estimate Ψ for $H_\nu = \nu(-\Delta)^r + \vec{u}\cdot\nabla$ on $L^2(\Omega)$.

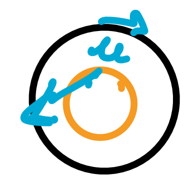
Example 1: Circularly symmetric and pipe parallel flows

Show enhanced dissipation when $\vec{u}$ has a certain symmetry in 2 and 3 space dimensions:

I. 2D case: $\Omega = \mathbb{R}^2$, $\vec{u}$ steady circularly symmetric

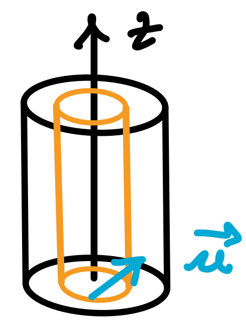
$$\boxed{\vec{u}(r,\theta) = r\,u(r)\,\vec{e}_\theta}\quad , r \geq 0,\ t \geq 0,$$

where $\vec{e}_\theta = (-\sin\theta, \cos\theta) \Rightarrow$ circular shear flow,



II. 3D case: $\Omega = \mathcal{D}(0,1) \times \mathbb{R}$ infinite, straight cylinder, $\vec{u}$ steady pipe parallel flow

$$\boxed{\vec{u}(r,\theta,z) = r\,u(r)\left(\sin(2\pi r)\,\vec{e}_\theta + \cos(2\pi r)\,\vec{e}_z\right.}$$



where $e_z = (0,0,1) \Rightarrow$ akin to Poiseuille flow

(r,θ) polar, (r,θ,z) cylindrical coordinates.

Conditions on the velocity profile $u(r)$

- To apply Resolvent estimates, make assumptions on u in both case I & II.

Assumption 1 (2D): $\exists\, m, N \in \mathbb{N}$, $c_1 > 0$, $\delta_0 \in \mathbb{R}_+$, satisfying: $\forall \lambda \in \mathbb{R}$ and any $0 < \delta < \delta_0$, $\exists\, n \leq N$ and $r_1, \dots, r_n \in \mathbb{R}_+$ such that

$$|u(r) - \lambda| \geq c_1 \delta^m, \quad \forall\, |r - r_j| \geq \delta, \ \forall j = 1 \dots n.$$

Example: $u(r) = r^m$, (Coti Zelati & Dolce).

Assumption 2 (3D): $\exists\, m, N \in \mathbb{N}$, $c_1 > 0$, $\delta_0 \in \mathbb{R}_+$, satisfying: $\forall \alpha, \lambda \in \mathbb{R}$ and any $0 < \delta < \delta_0$, $\exists\, n \leq N$ and $r_1, \dots, r_n \in \mathbb{R}_+$ such that

$$|u(r) \sin(2\pi r + \alpha) - \lambda| \geq c_1 \delta^m, \quad |r - r_j| \geq \delta, \ \forall j = 1 \dots n$$

Example: $u(r) = \cos(2\pi r)$, $m = 2$ (c.f. Feng-Feng-Wang).

Some remarks

- Assumption 1 and 2 inspired by work of Coti Zelati & Gallay on Taylor dispersion for shear flows.

- If u satisfies assumption 1 & 2, it has a finite # of critical points up to order m.

- In 2D, $\vec{u}$ unbounded. In 3D, $\vec{u}$ is tangent to $\partial\Omega$. In both cases, $\vec{u}$ div. free and vanishes at $r=0$.
  $\Rightarrow$ in 3D impose periodic b.c in z and homogeneous Neumann b.c. on θ in (r,θ) (Dirichlet ok too).

- convenient to apply the Fourier transform in θ, z:

$$\boxed{\begin{gathered} H_{\nu,k} := ik\,u(r) + \nu\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}\right)^{\gamma}, \quad k \in \mathbb{Z},\ \gamma = 1,2, \\ H_{\nu,k} := i\,u(r)\big(k_1 \sin(2\pi r) + k_2 \cos(2\pi r)\big) + \nu\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{k_1^2}{r^2} + k_2^2\right), \\ k = (k_1, k_2) \in \mathbb{Z}^2. \end{gathered}}$$

Main Results

Theorem 1 (2D): Let $\vec{u}$ circularly symmetric satisfy

Assumption 1. Let θ satisfy (ADE). $\exists$ $C_1, C_2 > 0$ such that

$$\|\theta_{\neq}(t)\|_{L^2(\mathbb{R}^2)} \leq C_1 e^{-C_2 \lambda_\nu t} \|(\theta_0)_{\neq}\|_{L^2(\mathbb{R}^2)}$$

where $\lambda_\nu = \nu^{\frac{m}{m+2\gamma}}$, $\theta_{\neq} := \theta - \int_0^{2\pi} \theta(r,\theta,t)\, d\theta$

Theorem 2 (3D): Let $\vec{u}$ parallel pipe flow satisfy

Assumption 2. Let θ satisfy (ADE). $\exists$ $C_1, C_2 > 0$ such that

$$\|\theta_{\neq}(t)\|_{L^2(\mathbb{R}^2)} \leq C_1 e^{-C_2 \lambda_\nu t} \|(\theta_0)_{\neq}\|_{L^2(\mathbb{R}^2)}$$

where $\lambda_\nu = \nu^{\frac{m}{m+2\gamma}}$, $\theta_{\neq} := \theta - \int_0^{2\pi} \int_0^{2\pi} \theta(r,\theta,z,t)\, d\theta\, dz$.

Note: $\theta_{\neq}$ is the projection onto $\operatorname{Ker}(H_\nu)^{\perp}$.

Sketch of proof :

① Because of Plancherel's identity enough to bound $e^{-tH_{\nu,k}}$, where $H_{\nu,k}$ operator on $L^2_r(\Omega) = L^2(\Omega, r\,dr\,d\theta\,dz)$

② m-accretivity of $H_{\nu,k}$ follows from m-accretivity of H_ν via isometric isomorphism + orthogonal projection.

③ Resolvent estimates follows by energy estimate

$\blacksquare$

Open Problems

- More examples of flows $\vec{u}$ in $\mathbb{R}^d$, $d = 1, 2$.
- Are Assumption 1 & 2 necessary?
- Applications to non-linear PDEs.

Example 2 : Shear flows on torus

Let $\Omega = \mathbb{T}^2$ and $\vec{u}(x,y) = (u(y), 0)$ steady horizontal shear flow.

Apply again the Fourier transform in x $\Rightarrow$ apply the resolvent estimate to $H_{\nu,k} = \nu \Delta_k^2 + ik\, u(y)$, $\Delta_k = -k^2 + \partial_y^2$, on $\mathcal{H} = L^2([0, 2\pi]; dy)$ to bound $H_\nu = \Delta^2 + u(y)\partial_x$ on $L^2(\mathbb{T}^2)$.

The Assumption on the velocity profile becomes:

Assumption 3 (shear flow): $\exists$ $m, N \in \mathbb{N}$ and $\delta_0 \in (0, L_2)$ with the property that, for any $\lambda \in \mathbb{R}$ and any y $0 < \delta < \delta_0$, $\exists$ $n \leq N$ and points $y_1, \dots, y_n \in [0, L_2)$ such that

$$\boxed{|u(y) - \lambda| \geq c_1 \left(\frac{\delta}{L_2}\right)^m}, \quad |y - y_j| < \delta \quad \forall j \in \{1, \dots, d\}$$

Example: $u(y) = (\sin y)^m$

then, the resolvent estimate give the following result.

Corollary: Let P_k be the L^2 projection onto the k-th horizontal mode. Then, $\exists$ ε_0', independent of ν and k

$$\| e^{-H_\nu t} P_k \|_{op} \leq e^{-\varepsilon_0' \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}} t + \pi/2}$$

$$\Rightarrow \boxed{\| e^{-H_\nu t} \|_{op} \leq e^{-\lambda_\nu' t + \pi/2}}, \quad t > 0, \quad \boxed{\lambda_\nu' = \varepsilon_0' \nu^{\frac{m}{m+4}}}.$$

Remark: ① we could also treat the case of a channel with periodic boundary conditions in y, and Dirichlet or Neumann conditions on ∂, as for the disk or pipe.

② Feng-Feng-Wang considered certain types of parallel flows on $\mathbb{T}^3 = [0, L_1] \times [0, L_2] \times [0, L_3]$:

$$\vec{u}(x,y,z) = (u(y) \sin(2\pi y/L_3), u(y) \cos(2\pi y/L_3), 0)$$

Applications to the 2D Kuramoto–Sivashinky equation

- Model for long wave-length instability in dissipative systems (flame front propagation, combustion).

- Work on 2D torus $\mathbb{T}^2 = [0, L_1] \times [0, L_2]$

  scalar form $\quad \partial_t \phi + \Delta^2 \phi + \Delta \phi + |\nabla \phi^2| = 0$

  for $\phi : \mathbb{T}^2 \times [0, T] \to \mathbb{R}$ $\quad$ (KSE)

  vector form $\quad \partial_t u + \Delta^2 u + \Delta u + u \cdot \nabla u = 0$

  where $u = \nabla \phi$.

- $d = 1 \Rightarrow$ global existence by energy methods (Tadmor) as $\int_{\mathbb{R}} u^2 \partial_x u = 0$.

- $d \geq 2 \Rightarrow$ no known Lyapunov functions (growing modes if $L_i > 2\pi$, no max principle, no energy estimates)

Known results in dimension $d>1$ (not modified KSE)

Local well-posedness for $\phi_0 \in L^p$ (Biswas-Swanson)

Continuation criteria based on the L^2 norm (Bellout-Benachour-Titi, Feng-M., Stanislavova-Stefanov)

Analyticity and Gevrey regularity (with rough data) for $t>0$ (Ambrose-M., Biswas-Swanson, Stanislavova-Stefanov)

Attractor & determining modes assuming solution global ($\|\nabla\phi(t)\| \le C \ \forall t>0$) (Nicolaenko-Scheurer-Temam)

Global existence for thin or anisotropic domains (Benachour-Kukavica-Rusin-Ziane, Kukavica-Massatt, Sell-Taboada), small data and no growing modes (Ambrose-M., Feng-M.), with advection (Coti Zelati-Dolce-Feng-M., Feng-M.), 1 growing mode (Ambrose-M.)

Global existence for 2D KSE with advection

- Study 2D KSE with advection by a shear flow $\vec{u}$:

$$\partial_t \phi + \nu \Delta^2 \phi + \nu \Delta \phi + \nu |\nabla \phi|^2 + \vec{v} \cdot \nabla \phi = 0 \qquad \text{(AKSE)}$$

where $\vec{u} = A \vec{v}$, $\nu = A^{-1}$, $\vec{v}(x,y) = (u(y), 0)$.

- Advection has a large kernel $\Rightarrow$ no enhanced dissipation in the kernel $\Rightarrow$ separate evolution on the kernel.

- Given $g \in L^2(\mathbb{T}^2)$, denote:

$$\langle g \rangle (y) = \frac{1}{L_1} \int_0^{L_1} g(t, x, y)\, dx \,, \qquad g_{\neq}(x,y) = g(x,y) - \langle g \rangle (y).$$

$\langle g \rangle$ projection onto kernel of $u(y)\partial_x$.

$g_{\neq}$ projection onto orthogonal complement in L^2

- Refer to $\langle g \rangle$, $g_{\neq}$ as kernel, projected components.

- From (AKSE), if ϕ solution of (AKSE), $\langle\phi\rangle$ satisfies

$$\boxed{\partial_t \langle\phi\rangle + \frac{\nu}{2L_1}\int_0^{L_1} |\nabla \phi_{\neq} + \nabla\langle\phi\rangle|^2 \, dx + \nu\, \partial_y^4\langle\phi\rangle + \nu\, \partial_y^2\langle\phi\rangle = 0}$$

while $\phi_{\neq}$ satisfies:

$$\boxed{\begin{aligned}\partial_t\phi_{\neq} + u(y)\,\partial_x\phi_{\neq} + \nu\,\Delta^2\phi_{\neq} &= -\frac{\nu}{2}|\nabla\phi_{\neq}|^2 + \frac{\nu}{2}\langle|\nabla\phi_{\neq}|^2\rangle \\ &\quad - \nu\,\partial_y\phi_{\neq}\,\partial_y\langle\phi\rangle - \nu\,\Delta\phi_{\neq}\end{aligned}}$$

$\Rightarrow$ two equations coupled through $\partial_y\langle\phi\rangle$ $\Rightarrow$ set $\psi = \partial_y\langle\phi\rangle$ that satisfies:

$$\boxed{\partial_t\psi + \frac{\nu}{2L_1}\int_0^{L_1}\partial_y|\nabla\phi_{\neq}|^2 \, dx + \nu\,\psi\,\partial_y\psi + \nu\,\partial_y^4\psi + \nu\,\partial_y^2\psi = 0}$$

- An L^2 continuation principle holds for these equations.

Main result (Coti Zelati - Dolce - Feng - M. '22):

Let $\phi_0 \in L^2(\mathbb{T}^2)$. Let $u(y)$ satisfy Assumption 3. Then, $\exists$ $0 < \nu_0 < 1$ depending on $L_1, L_2, \|\phi_0\|_{L^2}$ such that for any $0 < \nu < \nu_0$, $\exists$ global-in-time weak solution of (AKSE) with data $\phi(0) = \phi_0$.

Theorem extends to shear profile u with a finite number of critical points of order $m \geq 2$, but the resolvent estimate yields a worse bound for semigroup $e^{-H_\nu t}$.

The parameter ν_0 depends on the rate at which $\nu/\lambda(\nu) \to 0$ as $\nu \to 0$.

Bootstrap

- global existence theorem based on a bootstrap argument (He–Bedrossian).

- Local existence theory implies for $t \geq 0$:

Bootstrap Assumptions:

① $\|\phi_{\neq}(t)\|_{L^2} \leq 8 e^{-\lambda_\nu t/4} \|\phi_{\neq}(0)\|_{L^2}$;

② $\nu \int_0^t \|\Delta \phi_{\neq}(s)\|\, ds \leq 4 \|\phi_{\neq}(0)\|_{L^2}^2$.

Let t_0 be the maximal time such that ①, ② holds on $[0, t_0]$. Then on $[0, t_0]$;

$$\|\psi(t)\|_{L^2_y}^2 + \nu \int_0^t \|\partial_y^2 \psi(s)\|_{L^2_y}^2 ds \leq C_1 \left(\|\phi_{\neq}(0)\|_{L^2} \|\psi(0)\|_{L^2_y}^2\right) e^{C \nu t}$$

For ν small, decay of the semigroup implies bootstrap.

Proof of main result

Lemma (Bootstrap estimates): If ν_0 small enough and $0 < \nu < \nu_0$, then for all $t \in [0, t_0]$:

① $\|\phi_{\neq}(t)\|_{L^2} \leq 4 e^{-\lambda \nu t/4} \|\phi_{\neq}(0)\|^2$;

② $\nu \int_0^t \|\Delta \phi_{\neq}(s)\|^2_{L^2} \leq 2 \|\phi_{\neq}(0)\|^2_{L^2}$.

Step 1: By continuation in L^2 and Lemma, $t_0 = \infty \Rightarrow$
$\phi_{\neq} \in L^\infty([0, t_0]; L^2(\mathbb{T}^2)) \cap L^2([0,\infty); H^2(\mathbb{T}^2))$.

Step 2: Hence $\psi = \partial_y \langle \phi \rangle \in L^\infty([0,T); L^2(\mathbb{T}^1)) \cap L^2([0,T]; H^2(\mathbb{T}^1)) \Rightarrow \bar{\Phi} \in L^\infty([0,T]), \forall\, 0 < T < \infty$.

Step 3: $\nabla^2 \varphi = \nabla^2 \varphi_t + \nabla^2 \phi \in L^2(\mathbb{T}^2)$,

Step 4: By Poincaré + triangle inequality, $\langle \phi \rangle \in L^\infty([0,T]; L^2)$
$\Rightarrow \phi = \phi_{\neq} + \langle \phi \rangle \in L^\infty([0,T]; L^2(\mathbb{R}))$.