Canonical directions on surfaces in $M^2(c) \times \mathbb{R}$

Marian Ioan Munteanu

Alexandru Ioan Cuza University of Iasi, Romania

Fulbright Visiting Scholar at Michigan State University

UNIVERSITY OF ARKANSAS

J. William Fulbright College of Arts and Sciences

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Canonical coordinates and principal directions

1. **The ambient space $\mathbb{M}^2(c) \times \mathbb{R}$**
   - Constant Angle Surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$
2. **Surfaces in $\mathbb{S}^2 \times \mathbb{R}$**
3. **Surfaces in $\mathbb{H}^2 \times \mathbb{R}$**
   - Minkowski model of $\mathbb{H}^2$
   - Minimality and Flatness
4. **Surfaces in Euclidean space $\mathbb{E}^3$**

The ambient space $\mathbb{M}^2(c) \times \mathbb{R}$

- $c = 1 \Rightarrow \mathbb{M}^2(c) = \mathbb{S}^2 \Rightarrow$ the ambient space $\mathbb{S}^2 \times \mathbb{R}$
- $c = -1 \Rightarrow \mathbb{M}^2(c) = \mathbb{H}^2 \Rightarrow$ the ambient space $\mathbb{H}^2 \times \mathbb{R}$
- $c = 0 \Rightarrow \mathbb{M}^2(c) = \mathbb{R}^2 \Rightarrow$ the ambient space $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$

B. Nelli, H. Rosenberg, *Minimal surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. **33** (2) (2002), 263–292.

H. Rosenberg, *Minimal surfaces in* $\mathbb{M}^2 \times \mathbb{R}$, Illinois J. Math. **46** (4) (2002), 1177–1195.

Problem 1: Constant Angle Surfaces

A problem studied until now consists of the classification and characterization of Constant Angle Surfaces (CAS) in different ambient spaces. A CAS is an orientable surface whose unit normal makes a constant angle, denoted by θ, with a fixed direction.

The complete classification:

- F. Dillen, J. Fastenakels, J. Van der Veken, L. Vrancken, *Constant Angle Surfaces in* $\mathbb{S}^2 \times \mathbb{R}$, Monatsh. Math. **152** (2) (2007), 89–96.
- F. Dillen, M.I.M., *Constant Angle Surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. **40** (1) (2009), 85–97.
- M.I.M., A.I. Nistor, *A new approach on constant angle surfaces in* $\mathbb{E}^3$, Turkish J. Math. **33** (2) (2009), 169–178.
- Eugenio Garnica, Oscar Palmas, Gabriel Ruiz-Hernandez, *Hypersurfaces making a constant angle with a closed conformal vector field*, arXiv:1104.0050v1 [math.DG] (March 31, 2011).

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Problem 2: Canonical directions

When the ambient is of the form $\mathbb{M}^2 \times \mathbb{R}$, a favored direction is $\mathbb{R}$. It is known that for a constant angle surface in $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$, the projection of $\frac{\partial}{\partial t}$ (where t is the global parameter on $\mathbb{R}$) onto the tangent plane of the immersed surface, denoted by T, is a principal direction with the corresponding principal curvature identically zero.

Question

Study surfaces in $\mathbb{M}^2 \times \mathbb{R}$ such that T remains a principal direction but with the corresponding principal curvature different from 0.

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First answer in $\mathbb{S}^2 \times \mathbb{R}$

The characterization of surfaces with a principal direction:

Theorem (Dillen, Fastenakels, Van der Veken, 2009)

Let M be an immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ and p a point of M for which $\theta(p) \neq \{0, \frac{\pi}{2}\}$. Then T is a principal direction if and only if M considered as a surface in $\mathbb{E}^4$ is normally flat.

First answer in $\mathbb{S}^2 \times \mathbb{R}$

Proposition (classification result) - Dillen, Fastenakels, Van der Veken, 2009

A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a surface for which T is a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) in the neighborhood of a point p where $\theta(p) \notin \{0, \frac{\pi}{2}\}$ given by

$$F : M \to \mathbb{S}^2 \times \mathbb{R} : \ (x, y) \mapsto (F_1(x, y),\ F_2(x, y),\ F_3(x, y),\ F_4(x))$$

with

$$F_j(x, y) = \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv$$

for $j = 1, 2, 3$ where $\phi'(x) = \cos(\theta(x))$, $F_4'(x) = \sin(\theta(x))$, $(\alpha_1,\ \alpha_2,\ \alpha_3)$ is a curve in $\mathbb{S}^2$ and $F_1^2 + F_2^2 + F_3^2 = 1$. Moreover, α_1, α_2, α_3,ψ and ϕ are related by

$$\begin{aligned} \alpha_j'(y) \quad = \quad & -\cos(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \cos(\psi(x) + \phi(v)) dv \\ & -\sin(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv. \end{aligned}$$

General things in $\mathbb{H}^2 \times \mathbb{R}$

Notations:

- $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ the Riemannian product of $\left(\mathbb{H}^2(-1), g_H\right)$ and $\mathbb{R}$
- $\widetilde{g} = g_H + dt^2$ the product metric, t the (global) coordinate on $\mathbb{R}$
- $\widetilde{\nabla}$ the Levi Civita connection of $\widetilde{g}$
- $\partial_t := \frac{\partial}{\partial t}$ the unit vector field tangent to the $\mathbb{R}$-direction
- $\widetilde{R}$ either the curvature tensor $\widetilde{R}(X,Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$, or the

Riemann-Christoffel tensor on $\widetilde{M}$ defined by $\widetilde{R}(W,Z,X,Y) = \widetilde{g}(W, \widetilde{R}(X,Y)Z)$.

- $F : M \longrightarrow \widetilde{M}$ - isometric immersion ($\dim M = 2$)
- ξ - a unit normal vector to M, A - its shape operator
- $g = \widetilde{g}|_M$ - metric on M, ∇ - corresponding Levi Civita connection

(G) $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$, h the second fundamental form of M
(W) $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$

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Some useful formulas

Since $\partial_t := \frac{\partial}{\partial t}$ is of unit length, we decompose it as $\partial_t = T + \cos\theta\ \xi$ where

- T is the projection on $T(M)$ with $|T| = \sin\theta$ and
- θ is the angle function : $\cos\theta = \widetilde{g}(\partial_t, \xi)$.

(E.G.)

$$\begin{aligned} R(X,Y,Z,W) \quad = \quad & g(AX,W)g(AY,Z) - g(AX,Z)g(AY,W) - \\ & g(X,W)g(Y,Z) + g(X,Z)g(Y,W) + \\ & g(X,W)g(Y,T)g(Z,T) + g(Y,Z)g(X,T)g(W,T) - \\ & g(X,Z)g(Y,T)g(W,T) - g(Y,W)g(X,T)g(Z,T) \end{aligned}$$

(E.C.) $\quad (\nabla_X A)\, Y - (\nabla_Y A)\, X = \cos\theta\ (g(X,T)Y - g(Y,T)X)$

Computing the Gaussian curvature K, from the equation of Gauss it follows

$$K = \det A - \cos^2\theta.$$

Any vector field $X \in T(M)$ can be decomposed as $X = X_H + g(X,T)\partial_t$

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Proposition (Dillen, M., 2009)

For an arbitrary vector X tangent to M we have

$$\nabla_X T = \cos\theta AX \tag{1}$$

$$X(\cos\theta) = -g(AX, T). \tag{2}$$

For $\theta = const.$, eq. (2) yields $g(AT, X) = 0$, $\forall\ X \in T(M)$. Hence:

- if $T = 0$ on M, then ∂_t is always normal, so $M \subseteq \mathbb{H}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$.
- if $T \neq 0$ then T is a principal direction with principal curvature 0.

Question

Study surfaces in $\mathbb{H}^2 \times \mathbb{R}$ such that T remains a principal direction but with the corresponding principal curvature different from 0.

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First answers

Further on we suppose that θ **is different from** 0 **and** $\frac{\pi}{2}$.

Proposition (Dillen, M., Nistor, to appear Taiwanese J. Math., 2011)

If $\theta \neq 0, \frac{\pi}{2}$, then we can choose local coordinates (x, y) on the surface M isometrically immersed in $\widetilde{M}$ with ∂_x in the direction of T such that

$$g(x,y) = \frac{1}{\sin^2\theta} dx^2 + \beta^2(x,y) dy^2 \tag{3}$$

$$A = \begin{pmatrix} \theta_x \sin\theta & \theta_y \sin\theta \\ \frac{\theta_y}{\sin\theta\beta^2} & \frac{\sin^2\theta\beta_x}{\cos\theta\beta} \end{pmatrix} \tag{4}$$

and the functions θ and β are related by the PDE

$$\frac{\sin^2\theta}{\cos\theta}\frac{\beta_{xx}}{\beta} + \frac{\sin\theta\theta_x}{\cos^2\theta}\frac{\beta_x}{\beta} + \frac{\theta_y}{\sin\theta}\frac{\beta_y}{\beta^3} + \left(2\frac{\cos\theta\theta_y^2}{\sin^2\theta} - \frac{\theta_{yy}}{\sin\theta}\right)\frac{1}{\beta^2} - \cos\theta = 0. \tag{5}$$

An analogue result formulated for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ having T as principal direction, is the following

Proposition (Dillen, M., Nistor, 2011)

Let M be isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ with T a principal direction. Then, we can choose the local coordinates $(x,\ y)$ such that ∂_x is in the direction of T,

$$g = dx^2 + \beta^2(x,\ y)dy^2 \tag{6}$$

$$A = \begin{pmatrix} \theta_x & 0 \\ 0 & \tan\theta\frac{\beta_x}{\beta} \end{pmatrix}. \tag{7}$$

Moreover, the functions θ and β are related by the PDE

$$\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0 \tag{8}$$

and $\theta_y = 0$.

Canonical coordinates

Remark

For every two functions θ and β defined on a smooth simply connected surface M such that $\theta_y = 0$ and $\beta_{xx} + \tan\theta\theta_x\beta_x - \beta\cos^2\theta = 0$ for certain coordinates (x, y), we can construct an isometric immersion $F : M \to \mathbb{H}^2 \times \mathbb{R}$ with the shape operator (7) and such that it has a canonical principal direction.

Remark

Let M be an isometrically immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ such that T is a principal direction. The coordinates (x, y) on M such that ∂_x is collinear with T and the metric g has the form $g = dx^2 + \beta^2(x, y)dy^2$ are called *canonical coordinates*. Of course, they are not unique. More precisely, if (x, y) and $(\overline{x}, \overline{y})$ are both canonical coordinates, then they are related by $\overline{x} = \pm\, x + c$ and $\overline{y} = \overline{y}(y)$, where c is a real constant.

Minkowski model of the hyperbolic plane $\mathbb{H}^2$

Models for the hyperbolic plane:

1. the Klein model
2. the Poincaré disk
3. the upper half plane $\mathbb{H}^+$
4. Minkowski model $\mathcal{H}$

$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid \ x_1^2 + x_2^2 - x_3^2 = -1, \ x_3 > 0\}$$

with Lorentzian metric

$$\langle \ , \ \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

having constant Gaussian curvature -1.

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Characterization theorem

In order to study under which conditions T is a canonical principal direction, we regard the surface M as a surface immersed in $\mathbb{R}^3_1 \times \mathbb{R}$ (also denoted $\mathbb{R}^4_1$) having codimension 2.
The metric on the ambient space is given by $\widetilde{g} = dx_1^2 + dx_2^2 - dx_3^2 + dt^2$.
M is given by the immersion $F : M \to \mathbb{R}^3_1 \times \mathbb{R}$, $F = (F_1,\ F_2,\ F_3,\ F_4)$.

Theorem (Dillen, M., Nistor, 2011)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$. Then T is a principal direction if and only if M is normally flat in $\mathbb{R}^4_1$.

Classification theorem - version 1

Theorem (Dillen, M., Nistor, 2011)

If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \neq 0, \frac{\pi}{2}$, then T is a principal direction if and only if F is given, up to isometries of $\mathbb{H}^2 \times \mathbb{R}$, by

$$F(x,y) = (F_1(x,y),\ F_2(x,y),\ F_3(x,y),\ F_4(x))$$

with $F_j(x,y) = A_j(y)\sinh\phi(x) + B_j(y)\cosh\phi(x),\ j = \overline{1,3}$ and $F_4(x) = \int_0^x \sin\theta(\tau)d\tau$, where $\phi'(x) = \cos\theta$. The six functions A_j and B_j are found in one of the following cases

- **Case 1.**

$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{2j} \\ H_j'(y) &= B_j(y)\sinh\psi(y) - A_j(y)\cosh\psi(y) \end{aligned}$$

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- **Case 2.**

$$\begin{aligned} A_j(y) &= \int_0^y H_j(\tau)\sinh\psi(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)\cosh\psi(\tau)d\tau + c_{2j} \\ H_j'(y) &= -A_j(y)\sinh\psi(y) + B_j(y)\cosh\psi(y) \end{aligned}$$

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- **Case 3.**

$$\begin{aligned} A_j(y) &= \pm\int_0^y H_j(\tau)d\tau + c_{1j} \\ B_j(y) &= \int_0^y H_j(\tau)d\tau + c_{2j} \\ H_j'(y) &= c_{2j} \mp c_{1j} \end{aligned}$$

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where $H = (H_1, H_2, H_3)$ is a curve on the de Sitter space $\mathbb{S}^2_1$, ψ is a smooth function on M and $c_1 = (c_{11}, c_{12}, c_{13})$, $c_2 = (c_{21}, c_{22}, c_{23})$ are constant vectors.

$\square$

Clasiffication theorem - version 2

Theorem (Dillen, M., Nistor, 2011)

If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with angle function $\theta \neq 0, \frac{\pi}{2}$, then T is a principal direction if and only if F is given locally, up to isometries of the ambient space by

$$F(x, y) = (A(y) \sinh \phi(x) + B(y) \cosh \phi(x), \chi(x))$$

where $A(y)$ is a regular curve in $\mathbb{S}^2_1$, $B(y)$ is a regular curve in $\mathbb{H}^2$, such that $\langle A, B \rangle = 0$, $A' || B'$ and where $(\phi(x), \chi(x))$ is a regular curve in $\mathbb{R}^2$. The angle function θ of M depends only on x and it coincides with the angle function of the curve (ϕ, χ). In particular, we may arc length reparametrize (ϕ, χ); then (x, y) are canonical coordinates and $\theta'(x) = \kappa(x)$, the curvature of (ϕ, χ).

$\square$

Clasiffication theorem - version 3

Theorem (Dillen, M., Nistor, 2011)

Let $F : M \to \mathbb{H}^2 \times \mathbb{R}$ be an isometrically immersed surface M in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M has T as a principal direction if and only if F is given, up to rigid motions of the ambient space, by

$$F(x,y) = \Big(f(y)\cosh\phi(x) + N_f(y)\sinh\phi(x), \chi(x)\Big) \tag{9}$$

where $f(y)$ is a regular curve in $\mathbb{H}^2$ and $N_f(y) = \frac{f(y)\boxtimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$ represents the normal of f. Moreover, (ϕ, χ) is a regular curve in $\mathbb{R}^2$ and the angle function θ of this curve is the same as the angle function of the surface parameterized by F.

□

Examples

Now, we would like to give some examples of surfaces that can be retrieved from the classification theorem. Let us consider first $\psi(y) = 0$ for all y in **Case 1**, getting

$$A_j(y) = \int_0^y H_j(\tau)d\tau + c_{1j},\ B_j(y) = c_{2j},\ H_j'(y) = -\int_0^y H_j(\tau)d\tau - c_{1j}.$$

The parametrization F in this case is given by

Example (rotational surface)

$$\begin{aligned} F(x,y) \ = \ & \Big(\sin y \sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \cos y \sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big), \\ & \cosh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \int_0^x \sin\theta(\tau)d\tau\Big). \end{aligned}$$

Examples

Concerning **Case 3** in classification theorem, let us choose for example $c_1 = (0,1,0)$, $c_2 = (0,0,1)$ and $c_3 = (1,0,0)$. The parametrization in this case is given by

Example

$$\begin{aligned} F(x,y) &= \Big(A(y)\sinh\Big(\int_0^x \cos\theta(\tau)d\tau\Big) + \\ &\quad B(y)\cosh\Big(\int_0^x \cos\theta(\tau)d\tau\Big),\ \int_0^x \sin\theta(\tau)d\tau\Big) \end{aligned}$$

where $A(y) = \Big(y,\ 1-\frac{y^2}{2},\ \frac{y^2}{2}\Big)$ and $B(y) = \Big(y,\ -\frac{y^2}{2},\ 1+\frac{y^2}{2}\Big)$.

Examples

If $\theta(x) = x^2$, the surface is

Example

$$F(x,y) = \left(A(y)\sinh\left(\sqrt{\tfrac{\pi}{2}}\, C\left(\sqrt{\tfrac{2}{\pi}}\, x\right)\right) + B(y)\cosh\left(\sqrt{\tfrac{\pi}{2}}\, C\left(\sqrt{\tfrac{2}{\pi}}\, x\right)\right), \sqrt{\tfrac{\pi}{2}}\, S\left(\sqrt{\tfrac{2}{\pi}}\, x\right)\right)$$

where C and S are the traditional notations for the Fresnel integrals $C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$ respectively $S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$. The curve involved in the classification theorem is given in this case by $(\phi(x), \chi(x)) = (C(x),\ S(x))$, known as *Cornu spiral*.

Minimality

Theorem (Dillen, M., Nistor, 2011)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is minimal with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by
$F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}$

$$F(x,y) = \left(\frac{b(x)}{\sqrt{1+c_1^2-c_2^2}}\ ,\ \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}\ \sinh y,\ \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}\ \cosh y,\ \chi(x) \right)$$

$$F(x,y) = \left(\frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}}\ \cos y,\ \frac{\sqrt{a^2(x)+1}}{\sqrt{c_2^2-c_1^2-1}}\ \sin y,\ \frac{b(x)}{\sqrt{c_2^2-c_1^2-1}}\ ,\ \chi(x) \right)$$

$$F(x,y) = \left(b(x)\ y,\ \frac{b(x)}{2}\ (1-y^2) - \frac{1}{2b(x)},\ \frac{b(x)}{2}\ (1+y^2) + \frac{1}{2b(x)},\ \chi(x) \right)$$

Minimality

Theorem (cont.)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is minimal with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by

where

$$\chi(x) = \int_0^x \frac{1}{\sqrt{a^2(\tau)+1}}\, d\tau$$

with $a(x) = c_1 \cosh x + c_2 \sinh x$, $b(x) = a'(x)$ *and* $c_1, c_2 \in \mathbb{R}$.

Flatness

Theorem (Dillen, M., Nistor, 2011)

Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then M is flat with T as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by
$F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}$

$$F(x,\ y) = \left(\frac{x}{\sqrt{c+1}} \cos y,\ \frac{x}{\sqrt{c+1}} \sin y,\ \frac{\sqrt{x^2+c+1}}{\sqrt{c+1}},\ \chi(x) \right)$$

$$F(x,\ y) = \left(\frac{\sqrt{x^2+c+1}}{\sqrt{-c-1}},\ \frac{x}{\sqrt{-c-1}} \sinh y,\ \frac{x}{\sqrt{-c-1}} \cosh y,\ \chi(x) \right)$$

$$F(x,\ y) = \left(xy,\ \frac{x}{2}(1-y^2) - \frac{1}{2x},\ \frac{x}{2}(1+y^2) + \frac{1}{2x},\ \chi(x) \right)$$

where

$$\chi(x) = \int^x \frac{\sqrt{\tau^2+c}}{\sqrt{\tau^2+c+1}}\, d\tau,\ \ c \in \mathbb{R}.$$

The upper half plane model of $\mathbb{H}^2$

Method 1: Use **Cayley transformations** from $\mathcal{H}$ to H^+

$$x_1 = \frac{X}{Y} \qquad X = \frac{x_1}{x_3 - x_2}$$

$$x_2 = \frac{X^2 + Y^2 - 1}{2Y} \qquad Y = \frac{1}{x_3 - x_2}\ .$$

$$x_3 = \frac{X^2 + Y^2 + 1}{2Y}\ .$$

Method 2: Analytical approach - **solving the problem in $\mathbb{H}^+$** and then showing the consistence of results with $\mathcal{H}$:

A.I. Nistor, *On a class of surfaces in $\mathbb{H}^+ \times \mathbb{R}$*, preprint 2010.

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Canonical coordinates in $\mathbb{E}^3$

The characterization theorem:

Theorem (M., Nistor, 2011)

Let M be an isometrically immersed surface in $\mathbb{E}^3$. Let (x, y) be orthogonal coordinates on M such that T is collinear to ∂_x. Then, T is a principal direction on M everywhere if and only if $\theta_y = 0$.

M.I. Munteanu, A.I. Nistor, Complete classification of surfaces with a canonical principal direction in the Euclidean space $\mathbb{E}^3$, Cent. Eur. J. Math., 9 (2011), 2, 378–389.

Canonical coordinates in $\mathbb{E}^3$

The classification theorem:

Theorem (M., Nistor, 2011)

A surface M isometrically immersed in $\mathbb{E}^3$ with T a canonical principal direction is given (up to isometries of $\mathbb{E}^3$) by one of the following cases:

- **Case 1.**

$$F : M \to \mathbb{E}^3,\ F(x,\ y) = \left(\phi(x)(\cos y,\ \sin y) + \gamma(y),\ \int_0^x \sin\theta(\tau)d\tau\right)$$

where $$\gamma(y) = \left(-\int_0^y \psi(\tau)\sin\tau d\tau,\ \int_0^y \psi(\tau)\cos\tau d\tau\right)$$

- **Case 2. (Cylinders)**

$$F : M \to \mathbb{E}^3,\ F(x,\ y) = \left(\phi(x)\cos y_0, \phi(x)\sin y_0,\ \int_0^x \sin\theta(\tau)d\tau\right) + y\gamma_0$$

where $\gamma_0 = (-\sin y_0, \cos y_0, 0)$, $y_0 \in \mathbb{R}$, $\phi'(x) = \cos\theta$.

Canonical coordinates in $\mathbb{E}^3$ - minimality

Theorem (M., Nistor, 2011)

Let M be a surface isometrically immersed in $\mathbb{E}^3$. Then M is a minimal surface with T a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$$F : M \to \mathbb{E}^3$$

$$F(x,\ y) = \left(\sqrt{x^2+c^2}(\cos y,\ \sin y),\ \ln\left(x+\sqrt{x^2+c^2}\right)\right),\ c \in \mathbb{R}.$$

Remark

Moreover, we notice that this surface can be obtained rotating the catenary around the Oz-axis. Hence, we obtain that the only minimal surface in the Euclidean space with a canonical principal direction is the **catenoid**.

Canonical coordinates in $\mathbb{E}^3$ - flatness

Theorem (M., Nistor, 2011)

Let M be a surface isometrically immersed in $\mathbb{E}^3$. Then M is a flat surface with T a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$$F : M \to \mathbb{E}^3, \quad F(x,\ y) = \left(\phi(x)\cos y_0,\ \phi(x)\sin y_0,\ \int_0^x \sin\theta(\tau)d\tau\right) + y\gamma_0$$

where $\gamma_0 = \big(-\sin y_0, \cos y_0, 0\big),\ y_0 \in \mathbb{R}$.
Here $\phi(x)$ represents a primitive of $\cos\theta$.

Notice that this is **Case 2. (Cylinders)** from the classification theorem.

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