Objective Bayesian Analysis for Gaussian Hierarchical Models with Intrinsic Conditional Autoregressive Priors

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Spring Lecture Series – 2019
Motivating example

Hierarchical model with ICAR prior

Literature review

Sum-zero constrained ICAR prior

Objective priors

Simulation Study

Application: foreclosures and unemployment in OH in 2012

Conclusions
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Foreclosure rates

Unemployment rates
Motivating example

CAR Neighborhood Structure

[Diagram of a grid with some cells shaded]
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Hierarchical model with ICAR prior

\[ Y = X\beta + \theta + \phi, \]

where

- \( Y \) is the \( n \times 1 \) vector containing the response variable.
- \( X \) is a \( n \times p \) matrix of covariates.
- \( \beta \) is the \( p \times 1 \) vector of regression coefficients.
- \( \theta = (\theta_1, \ldots, \theta_n)' \) is a vector of unstructured random effects such that \( \theta_1, \ldots, \theta_n \iid N(0, \sigma^2) \).
- \( \phi = (\phi_1, \ldots, \phi_n)' \) is a vector of spatial random effects following a sum-zero constrained intrinsic CAR prior.
- \( \theta \) and \( \phi \) are assumed independent a priori.
To obtain a reference prior for a spatial hierarchical model, we have to first integrate out the spatial random effects.

This integral operation can only be performed when the spatial random effects have a well-defined distribution.

Usual improper intrinsic CAR random effects do not have a well-defined distribution.

That is probably the reason why a reference prior for this widely used spatial hierarchical model has not been published to date.
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Brief literature review

- CAR specifications for modeling areal data (Besag, 1974).
- Intrinsic CAR model as a prior for spatial random effects (Besag, York and Mollié, 1991).
- Comprehensive coverage of Gaussian Markov random fields (Rue and Held, 2005).
- Comprehensive coverage of hierarchical models for spatial data (Banerjee, Carlin and Gelfand, 2014).
Brief literature review

- OBayes for proper CAR to model observed areal data (Ferreira and De Oliveira, 2007; De Oliveira, 2012; Ren and Sun, 2013).

- OBayes analysis for geostatistical models (Berger et al, 2001; De Oliveira, 2007).

- OBayes for hierarchical models for areal data with proper CAR priors (Ren and Sun, 2014).
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CAR Neighborhood Structure
The intrinsic CAR model for a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is specified by its conditional distributions

$$p(\omega_i | \omega_{\sim i}) \propto \exp \left\{ -\frac{\tau \omega}{2} \left[ \sum_{i=1}^{n} \omega_i^2 h_i - 2 \sum_{i<j} \omega_i \omega_j g_{ij} \right] \right\},$$

where

- $\omega_{\sim i}$ is the vector of the CAR elements for all subregions except subregion $i$;
- $\tau \omega > 0$ is a precision parameter;
- $g_{ij} \geq 0$ is a measure of how similar subregions $i$ and $j$ are,
  
  $g_{ij} = g_{ji}$, and $h_i = \sum_{j=1}^{n} g_{ij}$. 

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Alternatively, we may write the joint density for $\omega$ as

$$p(\omega) \propto \exp \left\{ -\frac{\tau}{2} \omega^T H \omega \right\},$$

where $H$ is a symmetric, positive semi-definite precision matrix defined as

$$(H)_{ij} = \begin{cases} h_i, & \text{if } i = j \\ -g_{ij}, & \text{if } i \in N_j \\ 0, & \text{otherwise.} \end{cases}$$
Start with a vector of proper CAR random effects.

Project this vector of random effects onto the subspace of \( \mathbb{R}^n \) that is orthogonal to the subspace spanned by the vector \( n^{-1/2}1_n \).

Take the limit to get a sum-zero constrained ICAR.
We use a signal-to-noise ratio parametrization to express the proper CAR as

$$\phi^{**} \sim \mathcal{N} \left( 0, \frac{\sigma^2}{\tau_c} \Sigma_\lambda \right),$$

where

- $\sigma^2$ and $\tau_c > 0$ are unknown parameters,
- $\lambda > 0$ is a propriety parameter,
- $\Sigma_\lambda^{-1} = \lambda I_n + H$,
- When $\lambda \to 0$ we get the improper intrinsic CAR.
We define $\phi^* = P\phi^{**}$, where $P = (I_n - n^{-1}1_n1_n^T)$ is a centering (projection) matrix.

The distribution of $\phi^*$ is given by

$$\phi^* \sim N\left(0, \frac{\sigma^2}{\tau_c} \Sigma_{\phi\lambda}\right),$$

where $\Sigma_{\phi\lambda} = P\Sigma_{\lambda}P^T$.

By construction, $\sum_{i=1}^n \phi_i^* = 0$. 

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Now we take the limit as $\lambda \to 0$ to obtain the final distribution for $\phi$ given by

$$\phi \sim N \left(0, \frac{\sigma^2}{\tau_c} \Sigma\phi\right),$$

Thus $\Sigma\phi = H^+$ is the Moore-Penrose generalized inverse of $H$. 

$\Sigma\phi = \lim_{\lambda \to 0} \Sigma\phi\lambda = QM Q^T$, 

$M = \text{diag} \left(d_1^{-1}, \ldots, d_{n-1}^{-1}, 0\right),$

$d_1 \geq \cdots \geq d_{n-1} > d_n = 0$ are the ordered eigenvalues of $H$. 

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Sum-zero constrained ICAR prior (Keefe et al, 2018)

\[ p(\phi) = (2\pi)^{(n-1)/2} \tau_c^{(n-1)/2} \left( \prod_{i=1}^{n-1} d_i \right)^{1/2} \exp \left\{ -\frac{\tau_c}{2\sigma^2} \phi' H \phi \right\} \mathbb{1}(\phi' 1 = 0) \]
Sum-zero constrained ICAR prior

Compare to the Improper Intrinsic CAR

\[ p(\omega) \propto \exp \left\{ -\frac{\tau}{2} \omega^T H \omega \right\} \]
Facts about the Sum-Zero Constrained ICAR

- Keefe et al. (2018) consider proper CAR models with \( \Sigma^{-1}_\lambda = \lambda K + H \).
- Let \( K \) be a symmetric positive semi-definite matrix for which the sum of its elements is positive.
- Keefe et al. (2018) shows that for any matrix \( K \) in this class, the sum-zero constrained intrinsic CAR model does not depend on \( K \) and is, therefore, unique.
Facts about the Sum-Zero Constrained ICAR

- The singular Gaussian distribution $N(0, \tau^{-1}H^+) \text{ is the stationary distribution of a one-at-a-time Gibbs sampler applied to the improper ICAR prior with centering on the fly (Ferreira, 20XXa).}$

- Centering spatial random effects $\omega$ simulated from the full conditional distribution implied by the improper ICAR is equivalent to simulating from the full conditional distribution for $\phi$ implied by the $N(0, \tau^{-1}H^+)$ prior (Ferreira, 20XXb).

- Therefore, our reference prior is directly applicable to Gaussian hierarchical models with intrinsic CAR priors widely used in practice.
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All of the objective priors we have derived fall into a class of priors of the form

$$
\pi(\beta, \sigma^2, \tau_c) \propto \pi(\tau_c) \left(\frac{\tau_c}{\sigma^2}\right)^a,
$$

where $a \in \mathbb{R}$ is a hyperparameter and $\pi(\tau_c)$ is referred to as the marginal prior for $\tau_c$. 
Let $G^* = I_n - X(X'X)^{-1}X'$.

Let $Q^*$ be the matrix with columns that are the normalized eigenvectors corresponding to the non-zero eigenvalues of $G^*$.

Let $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-p} > 0$ be the ordered eigenvalues of $Q^*H^+Q^*$. 
The reference prior has $a = 1$ and

$$
\pi^R(\tau_c) \propto \frac{1}{\tau_c} \left[ \sum_{j=1}^{n-p} \left( \frac{\xi_j}{\tau_c + \xi_j} \right)^2 - \frac{1}{n-p} \left\{ \sum_{j=1}^{n-p} \left( \frac{\xi_j}{\tau_c + \xi_j} \right) \right\}^2 \right]^{1/2}.
$$

(Keefe, Ferreira, and Franck, 2019)
(i) The posterior $\pi^R(\beta, \sigma^2, \tau_c | y, X)$ resulting from the reference prior $\pi^R(\tau_c)$ is proper.

(ii) The $k^{th}$ moment of the marginal reference posterior $\pi^R(\tau_c | y, X)$ does not exist for $k \geq 1$.

(iii) The independence Jeffreys prior leads to an improper posterior distribution.

(iv) The Jeffreys-rule prior leads to an improper posterior distribution.
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Comparison of Priors

Competing priors:

- CARBayes package (version 4.0) has implemented gamma(0.001, 0.001) prior distributions as the default for both precision parameters.
- Best et al (1999) has used gamma(0.001, 0.001) and gamma(0.1, 0.1) prior distributions for the precisions of the unstructured and spatial random effects, respectively.

Performance is assessed using:

- frequentist coverage of credible interval.
- average interval length (IL).
- mean squared error (MSE).
Comparison of Priors

![Graph showing the comparison of different priors.

- Reference
- CARBayes
- NB

Graph axes are labeled as follows:
- Y-axis: $\log_{10}(\tau_c)$
- X-axis: $\log_{10}(\tau_c)$

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KL divergence between the spatial model and the independent model
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Simulation Design

- Square regions with sample sizes $n = 5^2, 7^2, 10^2$.
- $\sigma^2 = 2$.
- $\tau_c = 0.01, 0.032, 0.1, 0.32, 1, 3.2, 10$.
- First- and second-order neighborhood structure.
- $p = 1$ (intercept only) with $\beta = 1$; and $p = 6$ with $\beta = (-3, -2, -1, 1, 2, 3)'$.
- All covariates are generated from a normal distribution with mean 0 and variance 1.
- We generate results based on 1,000 simulated data sets for each combination of these levels of $n$, $\tau_c$, neighborhood structure, and $p$. 

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Frequentist coverage and mean interval length for $\tau_c$

n=100

n=49

n=25

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MSE for the posterior median of $\tau_c$

(a) $n = 100$

(b) $n = 49$

(c) $n = 25$

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One simulated dataset

- Dataset simulated with a first-order neighborhood structure, square regular grid, $n = 49$, and parameters $\beta = 1$, $\sigma^2 = 2$, and $\tau_c = 0.1$.
- The posterior medians of $\tau_c$ are equal to 1.0, 2.2, and 3.2 for the reference, NB, and CARBayes analyses, respectively.
- Contours correspond to HPD regions with credible levels equal to 10%, 20%, . . . , 90%.
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One simulated dataset

(a) Integrated likelihood

(b) Reference posterior

(c) NB posterior

(d) CARBayes posterior
We have found empirically that the probability of such a bad integrated likelihood behavior increases as the Moran-I statistic decreases.

For this particular dataset, the Moran-I statistic is equal to 0.203 with a corresponding p-value of 0.0181 (null hypothesis is of no spatial dependence).

This is not an extreme dataset; under the conditions used to simulate this dataset, the probability of the Moran-I statistic being less than 0.203 is about 0.12.
Another simulated dataset

- We now consider a dataset simulated under the same conditions as the previous dataset, but with a Moran-I statistic of moderate size.

- Specifically, for this dataset the Moran-I statistic is equal to 0.363 with a corresponding p-value of 0.00017 (null hypothesis is of no spatial dependence).

- This Moran-I statistic is close to the median of the sampling distribution of the Moran-I statistics.
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Another simulated dataset

(a) Integrated likelihood

(b) Reference posterior

(c) NB posterior

(d) CARBayes posterior

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We consider a data set containing foreclosure rates as a proportion of all housing transactions for each of the 88 counties in the state of Ohio for the year 2012.

Let $\text{SMR}_i = \frac{O_i}{E_i}$, where $O_i$ is the observed number of foreclosures in county $i$ and $E_i$ is the expected number of foreclosures in county $i$.

The expected counts are calculated by

$$E_i = n_i \left( \frac{\sum_j O_j}{\sum_j n_j} \right),$$

where $n_i$ is the total number of housing transactions in county $i$.

We consider $y_i = \log(\text{SMR}_i)$ as the response variable and unemployment rate as a covariate.
Foreclosures and unemployment in OH in 2012

SMRs

unemployment

Posterior median

Posterior s.d.
## Foreclosures and unemployment in OH in 2012

<table>
<thead>
<tr>
<th>Prior</th>
<th>Parameter</th>
<th>Estimate</th>
<th>95% Credible Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>$\beta_0$ (intercept)</td>
<td>-0.5561</td>
<td>(-0.9388, -0.1710)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$ (unemployment rate)</td>
<td>0.0488</td>
<td>(0.0008, 0.0967)</td>
</tr>
<tr>
<td></td>
<td>$\tau_c$</td>
<td>0.2519</td>
<td>(0.0017, 0.9781)</td>
</tr>
<tr>
<td></td>
<td>$\sigma^2$</td>
<td>0.0432</td>
<td>(0.0046, 0.0854)</td>
</tr>
<tr>
<td>CARBayes</td>
<td>$\beta_0$ (intercept)</td>
<td>-0.5612</td>
<td>(-0.9482, -0.1708)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$ (unemployment rate)</td>
<td>0.0496</td>
<td>(0.0009, 0.0977)</td>
</tr>
<tr>
<td></td>
<td>$\tau_c$</td>
<td>0.1959</td>
<td>(0.0009, 1.0231)</td>
</tr>
<tr>
<td></td>
<td>$\sigma^2$</td>
<td>0.0385</td>
<td>(0.0013, 0.0819)</td>
</tr>
<tr>
<td>NB</td>
<td>$\beta_0$ (intercept)</td>
<td>-0.5687</td>
<td>(-0.9516, -0.1802)</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$ (unemployment rate)</td>
<td>0.0504</td>
<td>(0.0020, 0.0985)</td>
</tr>
<tr>
<td></td>
<td>$\tau_c$</td>
<td>0.1589</td>
<td>(0.0010, 0.6290)</td>
</tr>
<tr>
<td></td>
<td>$\sigma^2$</td>
<td>0.0345</td>
<td>(0.0019, 0.0755)</td>
</tr>
</tbody>
</table>
Table: Foreclosures and unemployment in OH in 2012 – FBF-based posterior model probabilities

<table>
<thead>
<tr>
<th>Model</th>
<th>Posterior Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial Model intercept only</td>
<td>0.3603</td>
</tr>
<tr>
<td>Spatial Model with unemployment rate</td>
<td>0.6372</td>
</tr>
<tr>
<td>Independent Model intercept only</td>
<td>0.0019</td>
</tr>
<tr>
<td>Independent Model with unemployment rate</td>
<td>0.0006</td>
</tr>
</tbody>
</table>
Say we want to identify counties with risk higher than predicted by the regressors.

A common decision rule would be to flag counties for which $P(\phi_i > 0|Y) > 0.95$.

For the OH dataset, the reference, CARBayes, and NB analyses identify 3, 4, and 9 counties respectively.
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- Objective Bayesian analysis for a widely used Gaussian hierarchical model with ICAR prior for spatial data.

- Compared to two commonly used priors, the reference prior leads to a combination of favorable frequentist coverage, average interval length, and mean squared error.

- R package ref.ICAR available on CRAN (Porter et al., 2019).

- The reference prior proposed here leads to a proper posterior distribution and lets the data speak for themselves.