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Remarkably, this is not always the case.

The Principle of Symmetric Criticality holds if, for a given group action, the group invariant solutions for the Euler-Lagrange equations of any invariant Lagrangian are the Euler-Lagrange equations of a reduced Lagrangian.
PSC

Symmetric critical points are critical points of symmetric variations
In this talk I shall:

– review the known obstructions to the PSC.

– discuss the invariance of the boundary term in the first variational formula and its potential impact on PSC.
References


Reduction of Differential Equations

Let $G$ be a Lie group acting on a fiber bundle $E([x, u] \rightarrow [x])$. Then $G$ acts on the sections of $E(u = f(x))$. It is a symmetry group of a system of differential equations if the action of $G$ maps solutions to solutions.

Lie realized that the symmetry group can be computed and then used in different ways to solve the equation.

Group Invariant Solutions:

$$u_{xx} + u_{yy} = 0,$$

$$[x, y, u] \rightarrow [x \cos \theta + y \sin \theta, -y \sin \theta + x \cos \theta, u] \Rightarrow u = f(x^2 + y^2) \Rightarrow f'' + \frac{1}{r} f' = 0.$$  

Symmetry Reduction:

$$u'' = 0,$$

$$[x, y] \rightarrow [x + a, y] \Rightarrow v = u' \Rightarrow v' = 0.$$  

These 2 reduction processes are very different.
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These 2 reduction processes are very different.
Euler-Lagrange Equations

Recall a simple example of a problem in the calculus of variations:

\[ A[u] = \int_{\mathcal{R}} L(u, u_x, u_y) \, dx \, dy \quad u = u_0 \text{ on } \partial \mathcal{R} \]

\[ \frac{d}{dt} A[u + tv] |_{t=0} = \frac{d}{dt} \int_{\mathcal{R}} L(u + tv, u_x + tv_x, u_y + tv_y) \, dx \, dy |_{t=0} = \int_{\mathcal{R}} \left[ \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_y} v_y \right] \, dx \, dy + \int_{\partial \mathcal{R}} \left[ \frac{\partial L}{\partial u} u_x - \frac{\partial L}{\partial u_y} u_y \right] \, dy = \int_{\mathcal{R}} \{ \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_y} v_y \} \, dx \, dy + \int_{\partial \mathcal{R}} \frac{\partial L}{\partial u} u_x \, dy \]

This is the first variational formula – the RHS contains the Euler-Lagrange operator and the boundary term.
Euler-Lagrange Equations

Recall a simple example of a problem in the calculus of variations: Let $\mathcal{R}$ be a region in the $x$-$y$ plane. Find the functions $u = u(x, y)$ which minimize

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$$ \frac{d}{dt} A[u + tv] \bigg|_{t=0} $$

$$ = \frac{d}{dt} \iint_{\mathcal{R}} L(u + tv, u_x + tv_x, u_y + tv_y) \, dx \, dy \quad \text{at } t = 0 $$

$$ = \iint_{\mathcal{R}} \left[ \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_y} v_y \right] \, dx \, dy $$

$$ = \iint_{\mathcal{R}} v \left\{ \left[ \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u_x} - \frac{d}{dy} \frac{\partial L}{\partial u_y} \right] + \frac{d}{dx} \left[ \frac{\partial L}{\partial u_x} v + \frac{d}{dy} \frac{\partial L}{\partial u_y} v \right] \right\} \, dx \, dy $$

$$ = \iint_{\mathcal{R}} v \, EL(L) \, dx \, dy + \int_{\partial \mathcal{R}} v \left[ \frac{\partial L}{\partial u_x} \, dy - \frac{\partial L}{\partial u_y} \, dx \right] $$
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= \iint_{\mathcal{R}} v \left\{ \left[ \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u_x} - \frac{d}{dy} \frac{\partial L}{\partial u_y} \right] + \frac{d}{dx} \left[ \frac{\partial L}{\partial u_x} v + \frac{d}{dy} \frac{\partial L}{\partial u_y} v \right] \right\} \, dx \, dy
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This is the first variational formula – the RHS contains the Euler-Lagrange operator and the boundary term.
Reduction of Variational Principles and PSC

Let $G$ act on a fiber bundle $E([x, u] \rightarrow [x])$. Write the $G$ invariant sections as:

$$u(x) = \Phi(x, v(y)) \quad y = \varphi(x)$$

Let $L(x, u, \partial^k u)$ be a $G$ invariant Lagrangian. Let $\Delta = EL(L)$ be the Euler-Lagrangian equations.
Reduction of Variational Principles and PSC

Let $G$ act on a fiber bundle $E([x, u] 	o [x])$. Write the $G$ invariant sections as:

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Let $L(x, u, \partial^k u)$ be a $G$ invariant Lagrangian. Let $\Delta = \text{EL}(L)$ be the Euler-Lagrangian equations.

Consider the following diagram:

$$L(x, u, \partial^k u) \xrightarrow{\text{vary } u} \text{EL}(L) = \Delta(x, u, \partial^k u)$$

$$\Phi \downarrow \quad \Phi \downarrow \text{sym crt pt}$$

$$\tilde{L}(y, \nu, \partial^k \nu) \xrightarrow{\text{vary } \nu} \text{EL}(\tilde{L}) \quad \tilde{\Delta}(x, \nu, \partial^k \nu) .$$
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Example 1 - $SO(n)$ invariant solutions

Here we take the transformation group to be the standard action of $SO(3)$ on $\mathbb{R}^3$. The invariant sections are

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Our reduction diagram is:

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L = \frac{1}{2} (u_x^2 + u_y^2 + u_z^2) \quad \xrightarrow{\text{vary } u} \quad \text{EL}(L) = u_{xx} + u_{yy} + u_{zz}
\]

\[ u = v(r) \quad \xrightarrow{\text{sym crt pt}} \quad u = v(r) \]

\[ \tilde{L} = \frac{1}{2} r^2 (v')^2 \quad \xrightarrow{\text{sym var}} \quad v'' + \frac{2}{r} v' = v'' + \frac{2}{r} v' \]

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and PSC holds.
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**Theorem.** PSC always holds for the standard action of \( SO(n) \) on \( \mathbb{R}^n \times T^p(\mathbb{R}^n) \).
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and PSC holds.

**Theorem.** PSC always holds for the standard action of $SO(n)$ on $\mathbb{R}^n \times T^p(\mathbb{R}^n)$.

... but maybe not for $SO(n - 1, 1)$
Example 2 - Invariant solutions for a free action

We now consider the square of the Laplacian

\[ u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0 \]

and reduction by the 2 dim. non-abelian symmetry algebra

\[ Y = \partial_y, \quad S = x\partial_x + y\partial_y + z\partial_z + \frac{1}{2}u\partial_u \quad [Y, S] = Y. \]
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With \( u = \sqrt{(x)}v(s) \) and \( s = \frac{z}{x} \), the PDE becomes

\[ F = (s^2 + 1)^2v_{ssss} + 10s(s^2 + 1)v_{ss} + 15\frac{1}{2}v_{ss} - 15\frac{1}{16}v = 0. \]

(*\)

The Wilczynski invariant for this 4th order ODE

\[ W_4 = \partial F/\partial v_s + \frac{1}{2}d^2ds\partial F/\partial v_{sss} - \frac{1}{2}d^2ds\partial F/\partial v_{ss} - \frac{3}{4}\partial F/\partial v_{sss}d^2ds\partial F/\partial v_{sss} + \ldots \]

does not vanish and hence (*) is not variational.

Theorem. PSC always fails for reduction by a freely acting, non-unimodular symmetry group.
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\[ F = (s^2+1)^2 v_{ssss} + 10s(s^2+1)v_{sss} + \frac{15}{2} (3s^2+1)v_{ss} + \frac{15}{2} sv_s - \frac{15}{16} v = 0. \]

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The Wilczynski invariant for this 4th order ODE

\[ W_4 = \frac{\partial F}{\partial v_s} + \frac{1}{2} \frac{d^2 F}{ds^2} \frac{\partial F}{\partial v_{ss}} - \frac{1}{2} \frac{d F}{ds} \frac{\partial F}{\partial v_{ss}} - \frac{3}{4} \frac{\partial F}{\partial v_{ss}} \frac{d F}{ds} \frac{\partial F}{\partial v_{ss}} + \ldots \]

does not vanish
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We now consider the square of the Laplacian

\[ u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0 \]

and reduction by the 2 dim. non-abelian symmetry algebra

\[ Y = \partial_y, \quad S = x\partial_x + y\partial_y + z\partial_z, + \frac{1}{2}u\partial_u \quad [Y, S] = Y. \]

With \( u = \sqrt{(x)}v(s) \) and \( s = \frac{z}{x} \), the PDE becomes

\[ F = (s^2+1)^2v_{ssss} + 10s(s^2+1)v_{sss} + \frac{15}{2}(3s^2+1)v_{ss} + \frac{15}{2}sv_s - \frac{15}{16}v = 0. \]

The Wilczynski invariant for this 4th order ODE

\[ W_4 = \frac{\partial F}{\partial v_s} + \frac{1}{2} \frac{d^2}{ds^2} \frac{\partial F}{\partial v_{sss}} - \frac{1}{2} \frac{d}{ds} \frac{\partial F}{\partial v_{ss}} - \frac{3}{4} \frac{\partial F}{\partial v_{sss}} \frac{d}{ds} \frac{\partial F}{\partial v_{sss}} + \ldots \]

does not vanish and hence (*) is not variational.
Example 2 - Invariant solutions for a free action

We now consider the square of the Laplacian

\[ u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0 \]

and reduction by the 2 dim. non-abelian symmetry algebra

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**Theorem.** PSC always fails for reduction by a freely acting, non-unimodular symmetry group.
Example 3. Plane wave solutions in General Relativity

Our coordinates are $x, y, u, v$ and we consider the 5 dim symmetry algebra

$$\{ \partial_v, \partial_x, \partial_y, x\partial_v + P(u)\partial_x, y\partial_v + Q(u)\partial_v \}$$
Example 3. Plane wave solutions in General Relativity

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$$\{ \partial_v, \partial_x, \partial_y, x\partial_v + P(u)\partial_x, y\partial_v + Q(u)\partial_v \}$$

The general invariant metric is

$$g = A(u)du^2 + B(u)\gamma,$$

where

$$\gamma = -2du \, dv + \frac{dx^2}{P'(u)} + \frac{dy^2}{Q'(u)}.$$
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Here PSC fails (cataclysmically) –
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Here PSC fails (cataclysmically) –

The metric $g$ is not Ricci flat but every natural Lagrangian (constructed from the metric and covariant derivatives of curvature) vanishes identically on $g$. 

Abstract
References
1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary
4. A Brief History of PSC

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

1917: Weyl
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1972: MacCallum, Taub
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References
1. Reduction of DE
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   – Example 1
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4. History
5. PSC (Statement)
6. PSC (Details)
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8. Details
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10. Applications
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12. Powers of Laplacians
13. An Example
14. Summary

The geometric approach to PSC

Setup:

Let $\pi: E \to M$ be a fiber bundle with $G$ acting on $E, M$.

$G$ acts on sections of $E$ by $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$.

$q: M \to M/G$ is a smooth submersion.

Sufficient Conditions for PSC:

[I] Let $\Gamma$ be the inf. gen. for $G$ on $M$. The Lie algebra condition:

$H_{\text{top}}(\Gamma, G \times x) \neq 0$

[II] Let $V_p = \{ Z \in T_p E \mid \pi^*(Z) = 0 \}$. The Palais condition:

$[V^*_p] \cap \text{ann}(V_p) = 0$

[III] The boundary term in the first variational formula for the Lagrangian is $G$-invariant.
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Abstract
References
1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
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Back to the Examples

Example 1. Laplace’s equation in $\mathbb{R}^n$

Abstract
References
1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Back to the Examples

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- $H^n(\mathfrak{so}(n), \mathfrak{so}(n-1)) \neq 0$.
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- for a free action $\Gamma_x = 0$ and $H^{\text{top}}(\Gamma) = 0$, when $\Gamma$ is not unimodular.
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Example 2. Biharmonic equation
- for a free action $\Gamma_x = 0$ and $H^{\text{top}}(\Gamma) = 0$, when $\Gamma$ is not unimodular.

Example 3. Einstein equation
- the Palais condition always holds when there is a $G$ invariant, positive definite metric on $V_p$ but fails in the presence of null vectors.
Some Details

There are two fundamental constructions used to study PSC. First, it is necessary to have a good geometric characterization of the smooth invariant sections of \( E \).

Let

\[
\kappa(E) = \bigcup_{x \in M} \kappa_x(E) \quad \text{where} \quad \kappa_x(E) = \{ p \in E_x \mid g \cdot p = p \quad \text{for} \quad g \in G_x \}
\]

Every \( G \) invariant section of \( E \) takes values in \( \kappa(E) \).

We suppose that \( \kappa(E) \) is a fiber bundle. Then \( G \) acts regularly on \( \kappa(E) \) and we have:

\[
\tilde{E} = \kappa(E)/G \quad \overset{q_G}{\leftrightarrow} \quad \kappa(E) \quad \overset{t}{\longrightarrow} \quad E
\]

\[
\tilde{M} = M/G \quad \overset{q_G}{\leftrightarrow} \quad M \quad \overset{\text{id}}{\longrightarrow} \quad M.
\]

**Theorem** There is 1-1 correspondence between the \( G \) invariant sections of \( E \) and the sections of \( \tilde{E} \).
Second, it is necessary to have a good geometric characterization of the Euler-Lagrange operator.

To each fiber bundle $\pi : E \to M$, construct the infinity jet bundle $\pi^\infty : J^\infty(E) \to M$.

The contact distribution on $J^\infty(E)$ induces a direct sum decomposition (bicomplex)

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)) \quad \text{and} \quad d = d_H + d_V$$

Lagrangians are forms $\lambda \in \Omega^{n,0}(J^\infty(E))$ and the first variational formula is

$$d_V \lambda = EL(\lambda) + d_H(\Theta).$$

The fundamental lemma in the calculus of variations implies that this equation completely characterizes the Euler-Lagrange form $EL(\lambda)$.

This is precisely what is needed to study PSC.
In this jet space setting PSC becomes:

Find a chain $\chi \in \Lambda^p(TM)$ which induces a co-chain map between the bicomplexes $\Omega^*_G(J^\infty(E))$ and $\Omega^{(*,*)}(J^\infty(\tilde{E}))$.

If $\chi \rightarrow \lambda = q^*_G(\tilde{\lambda})$ then $E(\tilde{\lambda}) = q^*_G(\chi E(\lambda))$

PSC

The Euler-Lagrange form of the reduced Lagrangian is the reduction of the Euler-Lagrange form
Cochain maps on invariant de Rham complexes

The variational bicomplex gives a characterization of the E-L operator in terms of $d$. PSC then simplifies, partially, to:

Let $G$ act on $M$ with $p$-dim. orbits. Let $\pi: M \to \tilde{M} = M/G$. When is there a co-chain map $\Pi: \Omega^* G(M) \to \Omega^* - p(\tilde{M})$? Specifically, when does a multi-vector $\chi = X_1 \wedge X_2 \wedge \cdots \wedge X_p$ define a cochain map $\Pi(\omega) = \tilde{\omega}$ where $\chi \omega = \pi^*(\tilde{\omega})\omega \in \Omega^* G(M)$?

**Theorem.** There exists a multi-vector which defines a cochain map the $G$-invariant de Rham complex if and only if $H^p(g, Gx) \neq 0$. This gives $H^p(g, Gx) \neq 0$ as one of the sufficiency condition for PSC.
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Details

\[ \chi = X_1 \wedge X_2 \wedge \cdots X_p \]

\[ \chi \rightarrow \omega = \pi^*(\tilde{\omega}) \]

[i] - The form \( \chi \rightarrow \omega \) must be \( G \) basic.
[ii] - [i] implies the \( X_i \) must belong to \( \Gamma \), ie \( \chi \in \Lambda^p(\Gamma) \)
[iii] - [i] implies \( \chi \) must be \( G \) invariant
[iv] - [ii] and [iii] hold iff \( A^p(g, G_x) \neq 0 \)
[v] - \( \Pi \) commutes with \( d \) iff \( \mathcal{L}_R \chi = 0 \) for all \( G \) invariant vector fields \( R \).
[vi] - [v] holds iff all forms in \( A^{(p-1)}(g, G_x) \) are closed.

See IM, MF[4]
The first variational formula

\[ A[u] = \iint_{\mathcal{R}} L(u, u_x, u_y) \, dxdy. \]

\[ \frac{d}{dt} A[u + tv]_{t=0} = \iint_{\mathcal{R}} v \, EL(L) \, dxdy + \int_{\partial \mathcal{R}} v \left[ \frac{\partial L}{\partial u_x} \, dy - \frac{\partial L}{\partial u_y} \, dx \right] \]
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\[ \delta_v L = v \cdot EL(L) + D_i V^i \]

The boundary term is the \( n - 1 \) form

\[ \Theta = V^i \frac{\partial}{\partial x^i} - v, \quad v = dx^1 \wedge dx^2 \ldots dx^n \]
The first variational formula

\[ A[u] = \int \int_{\mathcal{R}} L(u, u_x, u_y) \, dx \, dy. \]

\[ \frac{d}{dt} A[u + tv] \big|_{t=0} = \int \int_{\mathcal{R}} v \, EL(L) \, dx \, dy + \int_{\partial \mathcal{R}} v \left[ \frac{\partial L}{\partial u_x} \, dy - \frac{\partial L}{\partial u_y} \, dx \right] \]

\[ \frac{d}{dt} L[u + th] \big|_{t=0} = \nabla \cdot EL(L) + D_x V^x + D_y V^y \]

\[ \delta_{\nu} L = \nabla \cdot EL(L) + D_i V^i \]

The boundary term is the \( n - 1 \) form

\[ \Theta = V^i \frac{\partial}{\partial x^i} - \nu, \quad \nu = dx^1 \wedge dx^2 \ldots dx^n \]

There is a general formula for \( \Theta \) but \( \Theta \) is not unique.
Applications of the boundary term

In many ways the boundary term in the 1st variational formula is just as important as the Euler-Lagrange term:
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For PSC we need to know that if $L$ is a $G$ Lagrangian, then a $G$ boundary term exists in the neighborhood of the jet of any invariant section.
Invariant Boundary Terms For Invariant Lagrangians

There are a number of theorems which insure the existence of an invariant boundary term (IBT):

- an IBT exists for Lagrangians of any order when \( n = 1 \);
- an IBT exists for Lagrangians of order 2;
- an IBT exists for generally covariant field theories such as general relativity;
- an IBT exists whenever the group action admits an invariant connection of \( TM \);
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Abstract
References
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2. COV
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   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary
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Conformally Groups and Powers of Laplacians

The group of conformal transformation on $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ does not admit an invariant connection of $T\mathbb{R}^n$.

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Using *DifferentialGeometry* in Maple, I did some symbolic experiments.

<table>
<thead>
<tr>
<th>$m \setminus n$</th>
<th>2</th>
<th>3</th>
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</table>
**Conjecture** The Lagrangians for the $\Delta^m(u)$ admit polynomial invariant boundary terms if and only if $w = m - n/2 \not< 0$. 
An Example \((m = 2)\)

Put

\[
\nu = dx^1 \wedge dx^2 \wedge \ldots dx^n, \quad \nu_j = \frac{\partial}{\partial x^j} \nu, \quad \nu_{ij} = \frac{\partial}{\partial x^j} \nu_i,
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The invariant Lagrangian is \(L = uu_{ij}^{ij} \nu\).
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This is \(O(n)\) invariant but not \(CO(n)\) invariant.

The possible correction terms are \(D(\eta)\), where

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If \(w = 0\) then the coefficient of \(b\) is \(CO(n)\) invariant.
The invariant sections are \( u = \text{constant} \).

Away from the points \( u_i = 0 \), we have the correction term

\[
\eta = \left[ \frac{D^i(N)v^j}{N} + \frac{D^i(N)u^j v}{N} \right] \nu_{ij} \quad N = g^{ij} u_i u_j
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The form $\eta$ is $O(n)$ invariant.

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**Remark.** Certain conformal invariants are called exceptional. The weights here are $w \neq 0$.

We have yet to determine if there are invariant boundary terms for these.
Summary

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- The Lie algebra cohomology condition and the Palais condition are well-understood.
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