Symmetric Criticality

Univ. of Arkansas Spring Lecture Series

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Abstract

Lie's method of solving PDE by looking for group invariant solutions is one of the most important general techniques for finding explicit solutions.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Lie's method of solving PDE by looking for group invariant solutions is one of the most important general techniques for finding explicit solutions.

When the PDE under consideration are the Euler-Lagrange equations derived from an invariant variational principle, one would hope that Lie's method can be used to determine a variational principle for the group invariant solutions.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Remarkably, this is not always the case.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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When the PDE under consideration are the Euler-Lagrange equations derived from an invariant variational principle, one would hope that Lie's method can be used to determine a variational principle for the group invariant solutions.

Remarkably, this is not always the case.

The Principle of Symmetric Criticality holds if, for a given group action, the group invariant solutions for the Euler-Lagrange equations of any invariant Lagrangian are the Euler-Lagrange equations of a reduced Lagrangian.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

PSC

Symmetric critical points are critical points of symmetric variations

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

PSC

Symmetric critical points are critical points of symmetric variations

In this talk I shall:

– review the known obstructions to the PSC.

– discuss the invariance of the boundary term in the first variational formula and its potential impact on PSC.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE
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3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Reduction of Differential Equations

Let G be a Lie group acting on a fiber bundle $E([x, u] \to [x])$.

Then G acts on the sections of $E(u = f(x))$. It is a symmetry group of a system of differential equations if the action of G maps solutions to solutions.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
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8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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Group Invariant Solutions:

$$u_{xx} + u_{yy} = 0, \quad [x,y,u] \to [x\cos\theta + y\sin\theta, -y\sin\theta + x\cos\theta, u]$$

$$\Longrightarrow \; u = f(x^2+y^2) \Longrightarrow \boxed{f'' + \frac{1}{r}f' = 0.}$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
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7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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Symmetry Reduction:

$$u'' = 0, \quad [x,y] \to [x+a, y]$$

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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These 2 reduction processes are very different.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Euler-Lagrange Equations

Recall a simple example of a problem in the calculus of variations:

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Euler-Lagrange Equations

Recall a simple example of a problem in the calculus of variations: Let $\mathcal{R}$ be a region in the x-y plane. Find the functions $u = u(x, y)$ which minimize

$$A[u] = \iint_{\mathcal{R}} L(u, u_x, u_y)\, dxdy \quad u = u_0 \text{ on } \partial R$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$A[u] = \iint_{\mathcal{R}} L(u, u_x, u_y)\, dxdy \quad u = u_0 \text{ on } \partial R$$

$$\frac{d}{dt} A[u + tv]_{|t=0}$$

$$= \frac{d}{dt} \iint_{\mathcal{R}} L(u + tv, u_x + tv_x, u_y + tv_y)\, dx\, dy \quad \text{at } t = 0$$

$$= \iint_{\mathcal{R}} \Big[\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_y} v_y\Big] dx\, dy$$

$$= \iint_{\mathcal{R}} v\Big\{\Big[\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u_x} - \frac{d}{dy}\frac{\partial L}{\partial u_y}\Big] + \frac{d}{dx}\Big[\frac{\partial L}{\partial u_x} v + \frac{d}{dy}\frac{\partial L}{\partial u_y} v\Big]\Big\} dx\, dy$$

$$= \boxed{\iint_{\mathcal{R}} v\, \mathrm{EL}(L) dx\, dy + \int_{\partial \mathcal{R}} v\Big[\frac{\partial L}{\partial u_x} dy - \frac{\partial L}{\partial u_y} dx\Big]}$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$= \boxed{\iint_{\mathcal{R}} v\, \mathrm{EL}(L) dx\, dy + \int_{\partial \mathcal{R}} v\Big[\frac{\partial L}{\partial u_x} dy - \frac{\partial L}{\partial u_y} dx\Big]}$$

This is the first variational formula – the RHS contains the Euler-Lagrange operator and the boundary term.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Reduction of Variational Principles and PSC

Let G act on a fiber bundle $E([x,u] \to [x])$. Write the G invariant sections as:

$$u(x) = \Phi(x, v(y)) \quad y = \varphi(x)$$

Let $L(x, u, \partial^k u)$ be a G invariant Lagrangian. Let $\Delta = \mathrm{EL}(L)$ be the Euler-Lagrangian equations.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Let $L(x, u, \partial^k u)$ be a G invariant Lagrangian. Let $\Delta = \mathrm{EL}(L)$ be the Euler-Lagrangian equations.

Consider the following diagram:

$$\begin{array}{ccc}
L(x,u,\partial^k u) & \xrightarrow{\text{vary } u} & \mathrm{EL}(L) = \Delta(x,u,\partial^k u) \\
\Phi \Big\downarrow & & \Phi \Big\downarrow \text{sym crt pt} \\
\tilde{L}(y,v,\partial^k v) & \xrightarrow[\text{sym var}]{\text{vary } v} & \mathrm{EL}(\tilde{L}) \stackrel{?}{=} \tilde{\Delta}(x,v,\partial^k v)\ .
\end{array}$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Consider the following diagram:

$$\begin{tikzcd}
L(x,u,\partial^k u) \arrow[r, "\text{vary } u"] \arrow[d, "\Phi"'] & \mathrm{EL}(L) = \Delta(x,u,\partial^k u) \arrow[d, "\Phi"', "\text{sym crt pt}"] \\
\tilde{L}(y,v,\partial^k v) \arrow[r, "\text{vary } v", "\text{sym var}"'] & \mathrm{EL}(\tilde{L}) \stackrel{?}{=} \tilde{\Delta}(x,v,\partial^k v) \ .
\end{tikzcd}$$

The Principle of Symmetric Criticality asserts that if the action of G satisfies certain conditions, then this diagram commutes.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Example 1 - $SO(n)$ invariant solutions

Here we take the transformation group to be the standard action of $SO(3)$ on $\mathbf{R}^3$. The invariant sections are

$$u = v(r), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Our reduction diagram is:

$$\begin{array}{ccc}
L = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2) & \xrightarrow{\text{vary } u} & \mathrm{EL}(L) = u_{xx} + u_{yy} + u_{zz} \\
\mathrm{u} = \mathrm{v}(\mathrm{r}) \Big\downarrow & & \mathrm{u} = \mathrm{v}(\mathrm{r}) \Big\downarrow \text{sym crt pt} \\
\tilde{L} = \frac{1}{2} r^2 (v')^2 & \xrightarrow[\text{sym var}]{\text{vary } v} & v'' + \frac{2}{r} v' \stackrel{?}{=} v'' + \frac{2}{r} v'
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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

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2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
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5. PSC (Statement)
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9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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and PSC holds.

Theorem. PSC always holds for the standard action of $SO(n)$ on $R^n \times T^p(R^n)$.

Abstract

References

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2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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...but maybe not for $SO(n-1, 1)$

Abstract

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2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
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5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Example 2 - Invariant solutions for a free action

We now consider the square of the Laplacian

$$u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0$$

and reduction by the 2 dim. non-abelian symmetry algebra

$$Y = \partial_y, \quad S = x\partial_x + y\partial_y + z\partial_z, + \frac{1}{2}u\partial_u \quad [Y, S] = Y.$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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With $u = \sqrt{(}x)v(s)$ and $s = \dfrac{z}{x}$, the PDE becomes

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Example 2 - Invariant solutions for a free action

We now consider the square of the Laplacian

$$u_{xxxx} + u_{yyyy} + u_{zzzz} + 2u_{xxyy} + 2u_{xxzz} + 2u_{yyzz} = 0$$

and reduction by the 2 dim. non-abelian symmetry algebra

$$Y = \partial_y, \quad S = x\partial_x + y\partial_y + z\partial_z, + \frac{1}{2}u\partial_u \quad [Y, S] = Y.$$

With $u = \sqrt{(}x)v(s)$ and $s = \dfrac{z}{x}$, the PDE becomes

$$F = (s^2+1)^2 v_{ssss} + 10s(s^2+1)v_{sss} + \frac{15}{2}(3s^2+1)v_{ss} + \frac{15}{2}sv_s - \frac{15}{16}v = 0. \quad (*)$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The Wilczynski invariant for this 4th order ODE

$$W_4 = \frac{\partial F}{\partial v_s} + \frac{1}{2}\frac{d^2}{ds^2}\frac{\partial F}{\partial v_{sss}} - \frac{1}{2}\frac{d}{ds}\frac{\partial F}{\partial v_{ss}} - \frac{3}{4}\frac{\partial F}{\partial v_{sss}}\frac{d}{ds}\frac{\partial F}{\partial v_{sss}} + \dots$$

does not vanish

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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does not vanish and hence (*) is not variational.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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does not vanish and hence (*) is not variational.

Theorem. PSC always fails for reduction by a freely acting, non-unimodular symmetry group.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Example 3. Plane wave solutions in General Relativity

Our coordinates are x, y, u, v and we consider the 5 dim symmetry algebra

$$\{\, \partial_v,\ \partial_x,\ \partial_y,\ x\partial_v + P(u)\partial_x,\ y\partial_v + Q(u)\partial_v \,\}$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$\{\, \partial_v,\ \partial_x,\ \partial_y,\ x\partial_v + P(u)\partial_x,\ y\partial_v + Q(u)\partial_v \,\}$$

The general invariant metric is

$$g = A(u)du^2 + B(u)\gamma, \quad \text{where}$$
$$\gamma = -2du\,dv + \frac{dx^2}{P'(u)} + \frac{dy^2}{Q'(u)}.$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Here PSC fails (cataclysmically) –

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The metric g is not Ricci flat but

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The general invariant metric is

$$g = A(u)du^2 + B(u)\gamma, \quad \text{where}$$
$$\gamma = -2du\,dv + \frac{dx^2}{P'(u)} + \frac{dy^2}{Q'(u)}.$$

Here PSC fails (cataclysmically) –

The metric g is not Ricci flat but every natural Lagrangian (constructed from the metric and covariant derivatives of curvature) vanishes identically on g.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

4. A Brief History of PSC

Abstract

References

1. Reduction of DE

2. COV

3. PSC

   – Example 1

   – Example 2

   – Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

1972: MacCallum, Taub

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

1972: MacCallum, Taub

1972: Lovelock

1979: Palais

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

1972: MacCallum, Taub

1972: Lovelock

1979: Palais

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

4. A Brief History of PSC

1917: Weyl

1969: Hawking

1972: MacCallum, Taub

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****: Dancer, Wang, Ziller *et al*

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

The geometric approach to PSC

Setup:

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

The geometric approach to PSC

Setup:

– Let $\pi\colon E \to M$ be a fiber bundle with G acting on E, M.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

The geometric approach to PSC

Setup:

- Let $\pi\colon E \to M$ be a fiber bundle with G acting on E, M.
- G acts on sections of E by $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Sufficient Conditions for PSC:

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Sufficient Conditions for PSC:

[I] Let Γ be the inf. gen. for G on M. The Lie algebra condition:

$$\boxed{H^{\text{top}}(\Gamma, G_x) \neq 0}$$

Abstract
References
1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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[I] Let Γ be the inf. gen. for G on M. The Lie algebra condition:

$$\boxed{H^{\text{top}}(\Gamma, G_x) \neq 0}$$

[II] Let $V_p = \{\, Z \in T_pE) \,|\, \pi_*(Z) = 0\}$. The Palais condition:

$$\boxed{[V_p^*]^{G_p} \cap \operatorname{ann}(V_p^{G_p}) = 0 \quad p = s(x_0), s \text{ invariant}}$$

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$\boxed{[V_p^*]^{G_p} \cap \operatorname{ann}(V_p^{G_p}) = 0 \quad p = s(x_0), s \text{ invariant}}$$

[III] The boundary term in the first variational formula for the Lagrangian is G invariant.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Back to the Examples

Example 1. Laplace's equation in $\mathbf{R}^n$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Back to the Examples

EXAMPLE 1. LAPLACE'S EQUATION IN $\mathbf{R}^n$

- $H^n(\mathfrak{so}(n), \mathfrak{so}(n-1) \neq 0$.

- the action of G is transverse to the fibers of E and the Palais conditions holds.

- invariant boundary terms always exist for linear actions.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

Back to the Examples

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EXAMPLE 2. BIHARMONIC EQUATION

- for a free action $\Gamma_x = 0$ and $H^{\mathrm{top}}(\Gamma) = 0$, when Γ is not unimodular.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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- for a free action $\Gamma_x = 0$ and $H^{\mathrm{top}}(\Gamma) = 0$, when Γ is not unimodular.

EXAMPLE 3. EINSTEIN EQUATION

- the Palais condition always holds when there is a G invariant, positive definite metric on V_p but fails in the presence of null vectors.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

Some Details

There are two fundamental constructions used to study PSC. First, it is necessary to have a good geometric characterization of the smooth invariant sections of E.
Let

$$\kappa(E) = \bigcup_{x \in M} \kappa_x(E) \text{ where } \kappa_x(E) = \{p \in E_x \mid g \cdot p = p \text{ for } g \in G_x\}$$

Every G invariant section of E takes values in $\kappa(E)$.

We suppose that $\kappa(E)$ is a fiber bundle. Then G acts regularly on $\kappa(E)$ and we have:

$$\begin{tikzcd}
\tilde{E} = \kappa(E)/G \arrow[d, "\pi_{\tilde{E}}"'] & \kappa(E) \arrow[l, "\mathbf{q}_G"'] \arrow[r, "\iota"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\
\tilde{M} = M/G & M \arrow[l, "\mathbf{q}_G"'] \arrow[r, "\mathrm{id}"] & M.
\end{tikzcd}$$

Theorem There is 1-1 correspondence between the G invariant sections of E and the sections of $\tilde{E}$.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Second, it is necessary to have a good geometric characterization of the Euler-Lagrange operator.

To each fiber bundle $\pi\colon E \to M$, construct the infinity jet bundle $\pi^\infty\colon J^\infty(E) \to M$.

The contact distribution on $J^\infty(E)$ induces a direct sum decomposition (bicomplex)

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)) \quad \text{and} \quad d = d_H + d_V$$

Lagrangians are forms $\lambda \in \Omega^{n,0}(J^\infty(E))$ and the first variational formula is

$$d_V\lambda = \mathrm{EL}(\lambda) + d_H(\Theta).$$

The fundamental lemma in the calculus of variations implies that this equation completely characterizes the Euler-Lagrange form $\mathrm{EL}(\lambda)$.

This is precisely what is needed to study PSC.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

In this jet space setting PSC becomes:

Find a chain $\chi \in \Lambda^p(TM)$ which induces a co-chain map between the bicomplexes $\Omega_G^{(*,*)}(J^\infty(E))$ and $\Omega^{(*,*)}(J^\infty(\tilde{E}))$.

$$\text{If} \quad \chi \lrcorner\, \lambda = \mathbf{q}_G^*(\tilde{\lambda}) \quad \text{then} \quad E(\tilde{\lambda}) = \mathbf{q}_G^*(\chi E(\lambda))$$

PSC

The Euler-Lagrange form of the reduced Lagrangian is the reduction of the Euler-Lagrange form

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Cochain maps on invariant de Rham complexes

The variational bicomplex gives a characterization of the E-L operator in terms of d. PSC then simplifies, partially, to:

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Let G act on M with p dim. orbits. Let $\pi : M \to \tilde{M} = M/G$.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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When is there a co-chain map $\Pi\colon \Omega^*_G(M) \to \Omega^{*-p}(\tilde{M})$?

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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When is there a co-chain map $\Pi\colon \Omega^*_G(M) \to \Omega^{*-p}(\tilde{M})$?

Specifically, when does a multi-vector $\chi = X_1 \wedge X_2 \wedge \cdots X_p$ define a co-chain map

$$\Pi(\omega) = \tilde{\omega} \quad \text{where} \quad \chi \lrcorner\, \omega = \pi^*(\tilde{\omega}) \quad \omega \in \Omega^*_G(M)$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$\Pi(\omega) = \tilde{\omega} \quad \text{where} \quad \chi \lrcorner\, \omega = \pi^*(\tilde{\omega}) \quad \omega \in \Omega^*_G(M)$$

Theorem. There exists a multi-vector which defines a cochain map the G invariant de Rham complex if and only if

$$H^p(\mathfrak{g}, G_x) \neq 0.$$

This gives $H^p(\mathfrak{g}, G_x) \neq 0$ as one of the sufficiency condition for PSC.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Details

$$\chi = X_1 \wedge X_2 \wedge \cdots X_p$$

$$\boxed{\chi \lrcorner\, \omega = \pi^*(\tilde{\omega})}$$

[i] - The form $\chi \lrcorner\, \omega$ must be G basic.

[ii] - [i] implies the X_i must belong to Γ, ie $\chi \in \Lambda^p(\mathbf{\Gamma})$

[iii] - [i] implies χ must be G invariant

[iv] - [ii] and [iii] hold iff $A^p(\mathfrak{g}, G_x) \neq 0$

[v] - Π commutes with d iff $\mathcal{L}_R \chi = 0$ for all G invariant vector fields R.

[vi] - [v] holds iff all forms in $A^{(p-1)}(\mathfrak{g}, G_x)$ are closed.

See IM, MF[4]

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

The first variational formula

$$A[u] = \iint_{\mathcal{R}} L(u, u_x, u_y)\, dxdy.$$

$$\frac{d}{dt} A[u + tv]_{|t=0} = \iint_{\mathcal{R}} v\, \mathrm{EL}(L) dxdy + \int_{\partial\mathcal{R}} v \Big[\frac{\partial L}{\partial u_x} dy - \frac{\partial L}{\partial u_y} dx \Big]$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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$$\frac{d}{dt} L[u + th]_{|t=0} = v \cdot EL(L) + D_x V^x + D_y V^y$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
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$$\frac{d}{dt} L[u + th]_{|t=0} = v \cdot EL(L) + D_x V^x + D_y V^y$$

$$\boxed{\delta_v L = v \cdot EL(L) + D_i V^i}$$

The boundary term is the $n-1$ form

$$\Theta = V^i \frac{\partial}{\partial x^i} \lrcorner\, \nu, \quad \nu = dx^1 \wedge dx^2 \ldots dx^n$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The boundary term is the $n-1$ form

$$\Theta = V^i \frac{\partial}{\partial x^i} \lrcorner\, \nu, \quad \nu = dx^1 \wedge dx^2 \ldots dx^n$$

There is a general formula for Θ but Θ is not unique.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Applications of the boundary term

In many ways the boundary term in the 1st variational formula is just as important as the Euler-Lagrange term:

Abstract

References

1. Reduction of DE

2. COV

3. PSC

   – Example 1

   – Example 2

   – Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

Applications of the boundary term

In many ways the boundary term in the 1st variational formula is just as important as the Euler-Lagrange term:

Abstract

References

1. Reduction of DE

2. COV

3. PSC

   – Example 1

   – Example 2

   – Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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- Noether's theorem

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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In many ways the boundary term in the 1st variational formula is just as important as the Euler-Lagrange term:

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Applications of the boundary term

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- Hamiltonian formalism
- variable endpoint problems

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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In many ways the boundary term in the 1st variational formula is just as important as the Euler-Lagrange term:

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- Hamiltonian formalism
- variable endpoint problems
- sufficiency conditions for minimum

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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- deformation invariants (Gauss-Bonnet-Cherm)

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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- deformation invariants (Gauss-Bonnet-Cherm)

For PSC we need to know that if L is a G Lagrangian. then a G boundary term exists in the neighborhood of the jet of any invariant section.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Invariant Boundary Terms For Invariant Lagrangians

There are a number of theorems which insure the existence of an invariant boundary term (IBT):

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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There are a number of theorems which insure the existence of an invariant boundary term (IBT):

- an IBT exists for Lagrangians of any order when $n = 1$;

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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Abstract

References

1. Reduction of DE
2. COV
3. PSC
   – Example 1
   – Example 2
   – Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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There are a number of theorems which insure the existence of an invariant boundary term (IBT):

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- an IBT exists for generally covariant field theories such as general relativity;

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

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Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

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- an IBT exists for Lagrangians of order 2;

- an IBT exists for generally covariant field theories such as general relativity;

- an IBT exists whenever the group action admits an invariant connection of TM;

- an IBT always exists locally in jet space, away from the jets of invariant sections;

Abstract

References

1. Reduction of DE

2. COV

3. PSC

– Example 1

– Example 2

– Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

Conformally Groups and Powers of Laplacians

The group of conformal transformation on $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ does not admit an invariant connection of $T\mathbf{R}^n$.

It is the simplest setting where one might find an invariant Lagrangian with no invariant boundary term.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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So we consider powers of the Laplacian

$$\Delta^m : \mathcal{E}[m - n/2] \to \mathcal{E}[-m - n/2]$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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$$\Delta^m : \mathcal{E}[m - n/2] \to \mathcal{E}[-m - n/2]$$

$$L = u\,\Delta^m(u)\,dx^1 \wedge dx^2 \ldots dx^n$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Conformally Groups and Powers of Laplacians

The group of conformal transformation on $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ does not admit an invariant connection of $T\mathbf{R}^n$.

It is the simplest setting where one might find an invariant Lagrangian with no invariant boundary term.

So we consider powers of the Laplacian

$$\Delta^m : \mathcal{E}[m - n/2] \to \mathcal{E}[-m - n/2]$$

$$L = u\,\Delta^m(u)\,dx^1 \wedge dx^2 \ldots dx^n$$

Using ***DifferentialGeometry*** in Maple, I did some symbolic experiments.

$m \backslash n$	2	3	4	5	6
1	Y	Y	Y	Y	Y
2	Y	Y	N	Y	Y
3	Y	Y	Y	Y	N

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Conjecture The Lagrangians for the $\Delta^m(u)$ admit polynomial invariant boundary terms if and only if $w = m - n/2 <> 0$.

Abstract

References

1. Reduction of DE

2. COV

3. PSC

   – Example 1

   – Example 2

   – Example 3.

4. History

5. PSC (Statement)

6. PSC (Details)

7. Cochain Maps

8. Details

9. The 1st var. formula

10. Applications

11. Inv. Boundary Terms

12. Powers of Laplacians

13. An Example

14. Summary

An Example ($m = 2$)

Put

$$\nu = dx^1 \wedge dx^2 \wedge \ldots dx^n, \; \nu_j = \frac{\partial}{\partial x^j} \lrcorner \, \nu, \; \nu_{ij} = \frac{\partial}{\partial x^j} \lrcorner \, \nu_i,$$

The invariant Lagrangian is $L = u u^{ij}{}_{ij} \, \nu$.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The invariant Lagrangian is $L = u u^{ij}{}_{ij} \, \nu$.

The boundary term given by the first variational formula is

$$\Theta = g^{(ij} g^{hk)} [u v_{jhk} - u_j v_{hk} + u_{jh} v_k + u_{jhk} v] \, \nu_i.$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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This is $O(n)$ invariant but not $CO(n)$ invariant.

The possible correction terms are $D(\eta)$, where

$$\eta = a \, u_i \, v_j \, dx^i \wedge dx^j + b \, u^i \, v^j \, \nu_{ij}.$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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If $w \neq 0$ then, there is choice of b for which $\Theta' = \Theta + D(\eta)$ is $CO(n)$ invariant.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The possible correction terms are $D(\eta)$, where

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If $w \neq 0$ then, there is choice of b for which $\Theta' = \Theta + D(\eta)$ is $CO(n)$ invariant.

If $w = 0$ then the coefficient of b is $CO(n)$ invariant.

Abstract
References
1. Reduction of DE
2. COV
3. PSC
– Example 1
– Example 2
– Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

The invariant sections are $u = constant$.

Away from the points $u_i = 0$, we have the correction term

$$\eta = [\frac{D^i(N)v^j}{N} + \frac{D^i(N)u^j v}{N}]\nu_{ij} \quad N = g^{ij}u_i u_j$$

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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The form η is $O(n)$ invariant.

It has the remarkable property that its Lie derivative with respect to an infinitesimal contact transformation is a polynomial in u and its derivatives.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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It has the remarkable property that its Lie derivative with respect to an infinitesimal contact transformation is a polynomial in u and its derivatives.

Remark. Certain conformal invariants are called exceptional. The weights here are $w \neq 0$.

We have yet to determine if there are invariant boundary terms for these.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Summary

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Summary

- The Principle of Symmetric Criticality is a powerful tool for finding exact solutions to non-linear differential equations arising in differential geometry.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Summary

- The Principle of Symmetric Criticality is a powerful tool for finding exact solutions to non-linear differential equations arising in differential geometry.

- A formulation of PSC in the language of jets leads to 3 obstructions.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

Summary

- The Principle of Symmetric Criticality is a powerful tool for finding exact solutions to non-linear differential equations arising in differential geometry.

- A formulation of PSC in the language of jets leads to 3 obstructions.

- The Lie algebra cohomology condition and the Palais condition are well-understood.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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- The Lie algebra cohomology condition and the Palais condition are well-understood.

- The invariant boundary term obstruction does not appear to arise as a practical matter.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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- Independent of PSC, the question of the existence of invariant boundary terms has been an open question for some time.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary

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- A formulation of PSC in the language of jets leads to 3 obstructions.

- The Lie algebra cohomology condition and the Palais condition are well-understood.

- The invariant boundary term obstruction does not appear to arise as a practical matter.

- Independent of PSC, the question of the existence of invariant boundary terms has been an open question for some time.

- The conformally invariant Laplacians in weight 0 give the first known examples of a variational problem with no invariant boundary term. This raises a number of interesting questions.

Abstract

References

1. Reduction of DE
2. COV
3. PSC
   - Example 1
   - Example 2
   - Example 3.
4. History
5. PSC (Statement)
6. PSC (Details)
7. Cochain Maps
8. Details
9. The 1st var. formula
10. Applications
11. Inv. Boundary Terms
12. Powers of Laplacians
13. An Example
14. Summary