How to recognize a conformally Kähler metric

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Let $(M, g)$ be a Riemannian four-manifold. Can you find a non–zero function $\Omega : M \rightarrow \mathbb{R}$ such that

\[
\hat{g} = \Omega^2 g
\]
is flat?

Answer: Need $\text{Weyl} = 0$.

Curvature decomposition: $\text{Riemann} = \text{Weyl} + \text{Ricci} + \text{scalar}$.

$\hat{g} = \Omega^2 g$ is Einstein?

Answer: ??? (but lots of necessary conditions are known: e.g. vanishing of the Bach tensor).

$\hat{g} = \Omega^2 g$ is Kähler?

1. $J : T M \rightarrow T M$ is a complex structure: $J^2 = -\text{Id}$ and $[T(1,0), T(1,0)] \subset T(1,0)$, where $T(1,0) = \{ X \in T M \otimes \mathbb{C}, J(X) = iX \}$.

2. $\hat{g}(X,Y) = \hat{g}(JX, JY)$ for any vector fields $X, Y$.

3. Fundamental two–form $\Sigma(X,Y) = \hat{g}(JX,Y)$ is closed.
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  3. Fundamental two–form $\Sigma(X, Y) = \hat{g}(JX, Y)$ is closed.
Always? No - let's count:

A metric in $2^n$ dimensions:

$$n(2^n + 1)$$

arbitrary functions of $2^n$ variables. Diffeomorphisms + conf. rescaling:

$$2^n - 2$$

The general Kähler metric can be locally described by the Kähler potential: there exists a function $K: M \to \mathbb{R}$ and a holomorphic coordinate system $(z_1, ..., z_n)$ such that

$$g = \partial^2 K / \partial z_j \partial z_k dz_j dz_k.$$ 

The difference between the number of arbitrary functions is $2^n - n - 2$, which is positive if $n > 1$ (every metric in 2D is Kähler).

Obstructions should be given by conformally invariant tensors.
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Degrees of freedom

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- Obstructions should be given by conformally invariant tensors.
Summary of results in four dimensions

- One–to–one correspondence between Kähler metrics in the conformal class of $g$ and parallel sections of a certain (canonical) connection $\mathcal{D}$ on a rank ten vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$. 

If the self–dual (SD) part of Weyl tensor $C^+$ of $g$ is non–zero we find the necessary and sufficient conditions: $C^+$ spinor must be of algebraic type $D$ and $T = 0$, $dV = 0$, where $T \in \Gamma(S \otimes \text{Sym}^5(S'))$, $V \in \Gamma(\Lambda^1(M))$ depend on $C^+$ and $\nabla C^+$. 

If $C^+ = 0$ we get some necessary conditions from the holonomy of the curvature of $\mathcal{D}$. E.g. A metric with $C^+ = 0$ is conformal to Einstein AND conformal to Kähler if and only if it admits an isometry.

Link with the metrisability problem (Bryant, MD, Eastwood. J. Diff. Geom. 2009): A projective structure on a surface $U$ is metrisable if and only if the induced (2, 2) conformal structure on $M = TU$ admits a Kähler metric or a para–Kähler metric.

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One-to-one correspondence between Kähler metrics in the conformal class of $g$ and parallel sections of a certain (canonical) connection $D$ on a rank ten vector bundle $E = \Lambda^2(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M)$.

If the self-dual (SD) part of Weyl tensor $C_+$ of $g$ is non-zero we find the necessary and sufficient conditions: $C_+$ spinor must be of algebraic type $D$ and

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Oriented Riemannian four-manifold \((M, g)\)

- \(\ast : \Lambda^2 \rightarrow \Lambda^2\), \(\ast^2 = \text{Id}\), \(\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-\).
Oriented Riemannian four-manifold \((M, g)\)

- \(* : \Lambda^2 \to \Lambda^2\), \(*^2 = \text{Id}\), \(\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-\).

- Riemann tensor gives rise to \(\mathcal{R} : \Lambda^2 \to \Lambda^2\).

\[
\mathcal{R} = \begin{pmatrix}
C_+ + \frac{R}{12} & \phi \\
\phi & C_- + \frac{R}{12}
\end{pmatrix}.
\]

\(C_\pm = \text{SD/ASD Weyl tensors}, \ \phi = \text{trace-free Ricci curvature}, \ \ R = \text{scalar curvature.}\)
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\[ g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2), \text{ where } v_1, v_2 \in \Gamma(S), w_1, w_2 \in \Gamma(S'). \]
Spinors in Four Dimensions

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- Two component spinor notation (love it or hate it):
  \( \mu \in \Gamma(S), \mu = \mu_A \). Spinor indices \( A, B, C, \cdots = 0, 1 \).
  \( \mu^A = \varepsilon^{AB} \mu_B, \mu_A = \mu^B \varepsilon_{BA} \). Metric \( g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'} \).
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- Spinors and self–duality. $\Sigma \in \Lambda^2(M), \Sigma_{ab} = \Sigma_{[ab]}$.
  \[ \Sigma_{AA'BB'} = \omega_{AB} \varepsilon_{A'B'} + \omega_{A'B'} \varepsilon_{AB}, \]
  where $\omega_{AB} = \omega_{(AB)}$ and $\omega_{A'B'} = \omega_{(A'B')}$. 

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- Spinors and self–duality. $\Sigma \in \Lambda^2(M)$, $\Sigma_{ab} = \Sigma_{[ab]}$.

$$\Sigma_{AA'B'B'} = \omega_{AB} \varepsilon_{A'B'} + \omega_{A'B'} \varepsilon_{AB},$$

where $\omega_{AB} = \omega_{(AB)}$ and $\omega_{A'B'} = \omega_{(A'B')}$. 
- Spinor curvature decomposition

$$R_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \psi_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD}$$
$$+ \phi_{ABC'D'} \varepsilon_{A'B'} \varepsilon_{CD} + \phi_{A'B'CD} \varepsilon_{AB} \varepsilon_{C'D'}$$
$$+ \frac{R}{12} \left( \varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'C'} \varepsilon_{B'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'D'} \varepsilon_{B'C'} \right).$$
Twistor equation

- $J^2 = -\text{Id}^2 \rightarrow \Sigma$ is SD or ASD. Make a choice:

$$\Sigma = \omega \otimes \varepsilon, \quad \omega \in \Gamma(S' \otimes S') \quad \text{(self–dual)}.$$
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- Conformal rescallings

\[ g \rightarrow \Omega^2 g, \quad \Sigma \rightarrow \Omega^3 \Sigma, \quad \text{so} \quad \omega \rightarrow \Omega^2 \omega. \]
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- **Lemma.** The metric \( g \) is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field \( \omega \in \Gamma(S' \otimes S') \) satisfying

\[ \nabla_A (A' \omega_{B'C'}) = 0, \quad (*) , \]

and such that \( |\omega|^2 \neq 0. \)
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- Conformal rescalings

$$g \rightarrow \Omega^2 g, \quad \Sigma \rightarrow \Omega^3 \Sigma, \quad \text{so} \quad \omega \rightarrow \Omega^2 \omega.$$

- Lemma. The metric $g$ is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field $\omega \in \Gamma(S' \otimes S')$ satisfying

$$\nabla_{A(A'} \omega_{B'C')} = 0, \quad (*) \quad (\nabla \omega = d\omega)$$

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**Twistor equation**

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- Conformal rescalling

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g \rightarrow \Omega^2 g, \quad \Sigma \rightarrow \Omega^3 \Sigma, \quad \text{so} \quad \omega \rightarrow \Omega^2 \omega.
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- **Lemma.** The metric \( g \) is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field \( \omega \in \Gamma(S' \otimes S') \) satisfying

\[
\nabla_A(A'\omega_{B'C'}) = 0, \quad (*) \quad (\nabla \omega = d\omega)
\]

and such that \( |\omega|^2 \neq 0 \).

- (*) is the (conformally invariant) twistor equation. Idea: prolong it, look for integrability conditions.
Prolongation of $\nabla_A (A' \omega_{B'C'}) = 0$

- Drop symmetrisation: $\nabla_{AA'} \omega_{B'C'} - \varepsilon_{A'B'} K_{C'A} - \varepsilon_{A'C'} K_{B'A} = 0$ for some $K \in \Lambda^1(M)$. 
Prolongation of $\nabla_A(A'\omega_{B'C'}) = 0$

- Drop symmetrisation: $\nabla_{AA'}\omega_{B'C'} - \varepsilon_{A'B'}K_{C'A} - \varepsilon_{A'C'}K_{B'A} = 0$ for some $K \in \Lambda^1(M)$.
- Differentiate and commute derivatives: $\psi^{E'}_{(A'B'C'\omega_{D'})E'} = 0$ and

$$\nabla_{AA'}K_{BB'} + P_{ABA'C'}\omega_{B'C'} - \varepsilon_{A'B'}\rho_{AB} = 0$$

(where $P_{ab} = (1/2)R_{ab} - (1/12)Rg_{ab}$) for some $\rho \in \Lambda^2_-(M)$.
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- Differentiate and commute derivatives:

$$\nabla_{AA'} \rho_{BC} - \omega_{A'E'} \nabla_{E'} \psi_{ABCD} + K_D^{A'} \psi_{ABCD} - 2P_A'E'A(BK_C)^{E'} = 0.$$
Prolongation of $\nabla_{A}(A'\omega_{B'C'}) = 0$

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- Now the system is closed: All derivatives of ‘unknowns’ have been determined.
Prolongation of $\nabla_A (A' \omega_{B'C'}) = 0$

- Drop symmetrisation: $\nabla_{AA'} \omega_{B'C'} - \varepsilon_{A'B'} K_{C'A} - \varepsilon_{A'C'} K_{B'A} = 0$ for some $K \in \Lambda^1(M)$.
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- Differentiate and commute derivatives:

$$\nabla_{AA'} \rho_{BC} - \omega_{A'E'} \nabla_{E'} D \psi_{ABCD} + K_{A'D} \psi_{ABCD} - 2 P_{A'E'A} (B K_{C'}^{E'}) = 0.$$  

- Now the system is closed: All derivatives of ‘unknowns’ have been determined.
- Geometric interpretation $\Psi = (\omega, K, \rho)$ is a section of a rank–10 vector bundle $E = \Lambda^2_{+}(M) \oplus \Lambda^1(M) \oplus \Lambda^2_{-}(M)$ which is parallel with respect to a connection $D$ determined by the blue equations.
Compact hyperbolic four manifold \((M, g)\). Weyl= 0, \(R = -1\). All local obstructions vanish.
Example: Local vs. Global obstructions

- Compact hyperbolic four manifold \((M, g)\). Weyl = 0, \(R = -1\). All local obstructions vanish.
- Assume globally defined non-degenerate \(\omega\) satisfies the twistor eq.

\[ \nabla \omega \neq 0 \] so \(\omega\) defines a Killing vector \(K^a\).

Killing identity \(\square K^a + R_{ab}K^b = 0\), where \(R_{ab} = -\frac{g_{ab}}{4}\).

Contract with \(K^a\), integrate by parts
\[ \int_M |\nabla K^a|^2 \sqrt{g} \, d^4x = -\frac{1}{4} \int_M |K^a|^2 \sqrt{g} \, d^4x. \]

Therefore \(K^a = 0\) and our assumption was wrong.
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- Assume globally defined non-degenerate $\omega$ satisfies the twistor eq.
  $\nabla \omega \neq 0$ so $\omega$ defines a Killing vector $K_a$.
- Killing identity $\Box K_a + R_{ab} K^b = 0$, where $R_{ab} = -g_{ab}/4$.
- Contract with $K^a$, integrate by parts

  \[ \int_M |\nabla K|^2 \sqrt{g} d^4x = -\frac{1}{4} \int_M |K|^2 \sqrt{g} d^4x. \]
Example: Local vs. Global obstructions

- Compact hyperbolic four manifold \((M, g)\). Weyl= 0, \(R = -1\). All local obstructions vanish.
- Assume globally defined non-degenerate \(\omega\) satisfies the twistor eq. \(\nabla \omega \neq 0\) so \(\omega\) defines a Killing vector \(K_a\).
- Killing identity \(\Box K_a + R_{ab} K^b = 0\), where \(R_{ab} = -g_{ab}/4\).
- Contract with \(K^a\), integrate by parts
  \[
  \int_M |\nabla K|^2 \sqrt{g} d^4x = -\frac{1}{4} \int_M |K|^2 \sqrt{g} d^4x.
  \]
- Therefore \(K^a = 0\) and our assumption was wrong.
Generic case $C_+ \neq 0$

Recall $\psi^{E'}_{(A'B'C'\omega D')}E' = 0$ (*).
**Generic case** $C_+ \neq 0$

- Recall $\psi_{(A'B'C'D')E'}^E = 0$ (*).
- Find that $C_+$ is of type $D$ i.e. $\psi_{A'B'C'D'} = \pm \omega_{(A'B'C'D')}$. 


**Generic case $C_+ \neq 0$**

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- Differentiate (*), impose the twistor equation.

**Theorem.** Let \((M, g)\) be a 4–manifold such that the self–dual part of the conformal curvature is non–zero. Then there exists a Kähler metric in \([g]\) if and only if \( C_+ \) is of type \( D \) and

\[
\nabla_A(A'\psi_{B'C'D'E'}) - V_A(A'\psi_{B'C'D'E'}) = 0, \quad \nabla_{[a}V_{b]} = 0,
\]

where \( V_{AA'} = \frac{1}{|\psi|^2} \left( \frac{1}{6} \nabla_{AA'}|\psi|^2 + \frac{4}{3} \psi_{B'C'D'E'} \nabla_{AB'} \psi_{C'D'E'A'} \right) \).
Theorem. Parallel sections $\Psi$ of $D$ on a rank 10 vector bundle $E \to M$ correspond to Kähler metrics in a conformal class.

Integrability conditions: $F \Psi = 0$ where $F = [D, D]$ (there are some indices, but let's not write them down).

If $F = 0$ then $g$ is conformally flat. Otherwise differentiate: $(DF) \Psi = 0$, $(DDF) \Psi = 0$, ...

After $K$ steps $F^K \Psi = 0$, where $F^K$ is a matrix of linear blue eqns.

Stop when rank $(F^K) = \text{rank}(F^{K+1})$. The space of parallel sections has dimension $(10 - \text{rank}(F^K))$.

Theorem. An anti-self–dual Einstein metric $g$ with $\Lambda \neq 0$ is conformal to a Kähler metric iff $g$ admits a Killing vector.

Examples of conf. classes with more than one (local) Kähler metrics: Fubini-Study metric on $\mathbb{CP}^2$ with reversed orientation.

Dunajski (DAMTP, Cambridge)
**Theorem.** Parallel sections $\Psi$ of $\mathcal{D}$ on a rank 10 vector bundle $E \to M$ correspond to Kähler metrics in a conformal class.

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If $\mathcal{F} = 0$ then $g$ is conformally flat. Otherwise differentiate:

- $(\mathcal{D}\mathcal{F})\Psi = 0$,
- $(\mathcal{D}\mathcal{D}\mathcal{F})\Psi = 0$,
- ...

After $K$ steps $\mathcal{F}_K\Psi = 0$, where $\mathcal{F}_K$ is a matrix of linear equations. Stop when rank $(\mathcal{F}_K) = \text{rank} (\mathcal{F}_{K+1})$. The space of parallel sections has dimension $(10 - \text{rank} (\mathcal{F}_K))$.

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for some one–form $\omega = \omega_i dx^i$. 

Theorem. The metric $g$ is conformal to (para) Kähler iff the projective structure is metrisable.
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**Bibliography**

- Walker (1953), Yano–Ishihara, ...
- Bryant, MD, Eastwood, 09.
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**Notes**

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Geodesics of a generic projective structures are not unparametrised geodesics of any metrics. The necessary and sufficient conditions for metrisability have recently been found (Bryant, MD, Eastwood, 09).
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Summary and Outlook


MD (2009) Solitons, Instantons and Twistors. OUP.

Conformal to Kahler
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Conformal to Kähler in dimensions four. Overdetermined system of linear PDEs. Necessary and sufficient conditions in the generic case. Prolongation bundle of rank 10 in the anti–self–dual case. Take this number, type it into Google, ….
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- Conformal to Kähler in dimensions four. Overdetermined system of linear PDEs. Necessary and sufficient conditions in the generic case. Prolongation bundle of rank 10 in the anti–self–dual case. Take this number, type it into Google, . . . .

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- General approach to overdetermined systems: prolong, construct connection, restrict its holonomy. Applicable to other ‘can you find’ problems.