How to recognize a conformally Kähler metric

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MD, Paul Tod `arXiv:0901.2261`, Mathematical Proceedings of the Cambridge Philosophical Society (2010).

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  3. Fundamental two–form $\Sigma(X, Y) = \hat{g}(JX, Y)$ is closed.

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- Obstructions should be given by conformally invariant tensors.

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- One–to–one correspondence between Kähler metrics in the conformal class of g and parallel sections of a certain (canonical) connection $\mathcal{D}$ on a rank ten vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$.

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- If the self–dual (SD) part of Weyl tensor C_+ of g is non–zero we find the necessary and sufficient conditions : C_+ spinor must be of algebraic type D and

$$T = 0, \qquad dV = 0,$$

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- Link with the metrisability problem (Bryant, MD, Eastwood. J. Diff. Geom. 2009): A projective structure on a surface U is metrisable if and only if the induced (2, 2) conformal structure on $M = TU$ admits a Kähler metric or a para–Kähler metric.

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Oriented Riemannian four-manifold (M, g)

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- Riemann tensor gives rise to $\mathcal{R} : \Lambda^2 \to \Lambda^2$.

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ + \frac{R}{12} & \phi \\ \hline \phi & C_- + \frac{R}{12} \end{array} \right).$$

$C_\pm =$ SD/ASD Weyl tensors, $\phi =$ trace-free Ricci curvature, $R =$scalar curvature.

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- Spinor curvature decomposition

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad + \phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &\quad + \frac{R}{12}(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}). \end{aligned}$$

Twistor equation

- $J^2 = -\mathrm{Id}^2 \to \Sigma$ is SD or ASD. Make a choice:

$$\Sigma = \omega \otimes \varepsilon, \quad \omega \in \Gamma(\mathbb{S}' \odot \mathbb{S}') \qquad \text{(self–dual).}$$

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- $(*)$ is the (conformally invariant) twistor equation. Idea: prolong it, look for integrability conditions.

Prolongation of $\nabla_{A(A'}\omega_{B'C')} = 0$

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- Geometric interpretation $\Psi = (\omega, K, \rho)$ is a section of a rank–10 vector bundle $E = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M)$ which is parallel with respect to a connection $\mathcal{D}$ determined by the blue equations.

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- Therefore $K^a = 0$ and our assumption was wrong.

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  Theorem. Let (M, g) be a 4–manifold such that the self–dual part of the conformal curvature is non–zero. Then there exists a Kähler metric in $[g]$ if and only if C_+ is of type D and

$$\nabla_{A(A'}\psi_{B'C'D'E')} - V_{A(A'}\psi_{B'C'D'E')} = 0, \qquad \nabla_{[a}V_{b]} = 0,$$

where $V_{AA'} = \frac{1}{|\psi|^2}\left(\frac{1}{6}\nabla_{AA'}|\psi|^2 + \frac{4}{3}\psi^{B'C'D'E'}\nabla_{AB'}\psi_{C'D'E'A'}\right)$.

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- Integrability conditions: $\mathcal{F}\Psi = 0$ where $\mathcal{F} = [\mathcal{D}, \mathcal{D}]$ (there are some indices, but lets not write them down).
- If $\mathcal{F} = 0$ then g is conformally flat. Otherwise differentiate: $(\mathcal{D}\mathcal{F})\Psi = 0, (\mathcal{D}\mathcal{D}\mathcal{F})\Psi = 0, \ldots$

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- Examples of conf. classes with more than one (local) Kähler metrics: Fubini-Study metric on $\mathbb{CP}^2$ with reversed orientation.

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- Theorem. The metric g is conformal to (para) Kähler iff the projective structure is metrisable.

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- General approach to overdetermined systems: prolong, construct connection, restrict its holonomy. Applicable to other 'can you find' problems.