Ambient Metrics and Exceptional Holonomy

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Holonomy

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$\mathrm{Hol}(M, g) = \{e\}$ if and only if g is flat.

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For many, but not all, groups on the list, examples were known of (M,g) with that holonomy.

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More such metrics are known now, but they are not so easy to come by. New examples are of interest.

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From now on, $G_2 = G_2^s$.

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So any diffeomorphism preserving $\mathcal{D}$ also preserves the $(2,3)$ conformal structure on $S^2 \times S^3$!

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For any $\mathcal{D}$, can choose local coordinates (x, y, z, p, q) on M so that

$$\mathcal{D} = \text{span}\{\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\}$$

where $F = F(x, y, z, p, q)$ and F_{qq} is nonvanishing.

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Approximately 70 terms. Very nasty.

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Given conformal manifold $(M, [g])$ of signature (p, q), $p + q = n$.

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$\mathcal{G} =$ null cone of $|x|^2 - |y|^2$

Leistner-Nurowski Result

Put these together:

$$\mathcal{D} \subset TM^5 \quad \overset{Nurowski}{\rightarrow} \quad (M, [g]) \quad \overset{Ambientmetric}{\rightarrow} \quad (\widetilde{\mathcal{G}}^7, \widetilde{g})$$

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But what about $\widetilde{g}$ for other $\mathcal{D}$?

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Work with Travis Willse.

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Theorem. (Hammerl-Sagerschnig, 2009) Nurowski's conformal structures $(M, [g])$ associated to generic $\mathcal{D}$ are characterized by the existence of a parallel tractor 3-form of split type.

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Immediately conclude $\mathrm{Hol}(\widetilde{\mathcal{G}}, \widetilde{g}) \subset G_2$.

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Say that $\mathcal{A}$ is 3-nondegenerate at p if the only vector $Y \in \mathcal{D}_p$ such that $\mathcal{A}(X, Y, Y, Y) = 0$ for all $X \in \mathcal{D}_p$ is $Y = 0$.

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Let $\mathcal{A} \in \Gamma(S^4 \mathcal{D}^*)$ be Cartan's fundamental curvature invariant for generic distributions $\mathcal{D} \subset M^5$. $\mathcal{A}$ is a binary quartic.

Say that $\mathcal{A}$ is 3-nondegenerate at p if the only vector $Y \in \mathcal{D}_p$ such that $\mathcal{A}(X, Y, Y, Y) = 0$ for all $X \in \mathcal{D}_p$ is $Y = 0$.

Theorem. Given $(M, \mathcal{D})$ real analytic. If there are p, $q \in M$ so that L_p is injective and $\mathcal{A}_q$ is 3-nondegenerate, then $\widetilde{g}$ has holonomy $= G_2$.

Conditions for $\mathrm{Hol}(\widetilde{\mathcal{G}}, \widetilde{g}) = G_2$

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This is a weak condition, explicitly checkable.