Ambient Metrics
and
Exceptional Holonomy

Robin Graham
University of Washington

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\(\text{Hol}(M, g) = \{e\}\) if and only if \(g\) is flat.
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**Question** becomes: Does every group on Berger’s list arise as a holonomy group?

For many, but not all, groups on the list, examples were known of $(M, g)$ with that holonomy.
Other than $SO_e(p, q)$, every group on Berger’s list occurs for $n$ even, with two exceptions.
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More such metrics are known now, but they are not so easy to come by. New examples are of interest.
Let $\varphi \in \Lambda^3 \mathbb{R}^7^*$. 
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$$(X \downarrow \varphi) \wedge (Y \downarrow \varphi) \wedge \varphi = \langle X, Y \rangle_{\varphi} e_1^* \wedge \ldots \wedge e_7^*$$
Let $\varphi \in \Lambda^3 \mathbb{R}^7^\ast$. Define $\langle \cdot, \cdot \rangle_{\varphi}$ by

$$(X \perp \varphi) \wedge (Y \perp \varphi) \wedge \varphi = \langle X, Y \rangle_{\varphi} e_1^* \wedge \ldots \wedge e_7^*$$

**Definition.** $\varphi$ is nondegenerate if $\langle X, Y \rangle_{\varphi}$ is nondegenerate.
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**Theorem.** $\varphi$ nondegenerate $\implies$ $\pm \langle \cdot, \cdot \rangle_{\varphi}$ has signature $(7, 0)$ or $(3, 4)$. 
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**Fact.** $\varphi^c$ and $\varphi^s$ are unique up to $GL(7, \mathbb{R})$. 
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G_2^c = \{ A \in GL(7, \mathbb{R}) : A^* \varphi^c = \varphi^c \} \subset SO(7)
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From now on, $G_2 = G_2^s$. 

$G_2$
2-plane Fields in Dimension 5

Let $\mathcal{D} \subset TM^5$, $\dim \mathcal{D}_x = 2$. 
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**Definition.** $\mathcal{D}$ is generic if $X, Y, Z, [X, Z], [Y, Z]$ are everywhere linearly independent.
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Recall $G_2 \subset SO(3, 4) = \text{conformal group of } (S^2 \times S^3, g_{S^2} - g_{S^3})$. 
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Recall $G_2 \subset SO(3, 4) = \text{conformal group of } (S^2 \times S^3, g_{S^2} - g_{S^3})$. The action of $G_2$ is the restriction of the conformal action of $SO(3, 4)$. 
2-plane Fields in Dimension 5

Let $D \subset TM^5$, $\dim D_x = 2$. $X$, $Y$ local frame. Set $Z = [X, Y]$.

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Recall $G_2 \subset SO(3, 4) = \text{conformal group of } (S^2 \times S^3, g_{S^2} - g_{S^3})$.

The action of $G_2$ is the restriction of the conformal action of $SO(3, 4)$.

So any diffeomorphism preserving $D$ also preserves the $(2, 3)$ conformal structure on $S^2 \times S^3$!
Theorem. (Nurowski, 2005) Any $\mathcal{D} \subset TM^5$ generic. There is a conformal class $[g]$ on $M$ of signature $(2, 3)$ associated to $\mathcal{D}$. 
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Follows immediately from the existence of the Cartan connection.

For any $\mathcal{D}$, can choose local coordinates $(x, y, z, p, q)$ on $M$ so that

$$\mathcal{D} = \text{span}\{\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\}$$

where $F = F(x, y, z, p, q)$ and $F_{qq}$ is nonvanishing.
**Theorem.** (Nurowski, 2005) Any $\mathcal{D} \subset TM^5$ generic. There is a conformal class $[g]$ on $M$ of signature $(2, 3)$ associated to $\mathcal{D}$.

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Nurowski gives a formula for $g$ in terms of $F$ and its derivatives of orders $\leq 4$.

Approximately 70 terms. Very nasty.
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Obtain a manifold \(\tilde{\mathcal{G}}\) of dimension \(n + 2\) with an embedded hypersurface \(\mathcal{G} \subset \tilde{\mathcal{G}}\).
Given conformal manifold \((M, [g])\) of signature \((p, q)\), \(p + q = n\). Obtain a manifold \(\tilde{G}\) of dimension \(n + 2\) with an embedded hypersurface \(G \subset \tilde{G}\) and a formal expansion along \(G\) for a metric \(\tilde{g}\) on \(\tilde{G}\) of signature \((p + 1, q + 1)\).
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Let \(\mathcal{G} = \{(x, g_x) : x \in M, g \in [g]\} \subset S^2 T^* M\). Metric bundle.
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Dilations \(\delta_s : G \to G\)

\[\delta_s(x, g_x) = (x, s^2 g_x)\]

Set \(\tilde{G} = G \times (-1, 1)\).
Given conformal manifold \((M, [g])\) of signature \((p, q)\), \(p + q = n\).

Obtain a manifold \(\mathcal{G}\) of dimension \(n + 2\) with an embedded hypersurface \(\mathcal{G} \subset \mathcal{G}\) and a formal expansion along \(\mathcal{G}\) for a metric \(\tilde{g}\) on \(\mathcal{G}\) of signature \((p + 1, q + 1)\).

Let \(\mathcal{G} = \{(x, g_x) : x \in M, g \in [g]\} \subset S^2 T^* M\). Metric bundle.

Dilations \(\delta_s : \mathcal{G} \rightarrow \mathcal{G}\) \(\delta_s(x, g_x) = (x, s^2 g_x)\)

Set \(\mathcal{G} = \mathcal{G} \times (-1, 1)\). Inclusion: \(\iota : \mathcal{G} \rightarrow \mathcal{G}, \quad \iota(z) = (z, 0)\).
Given conformal manifold \((M, [g])\) of signature \((p, q), p + q = n\).
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Inclusion: \(\iota : G \rightarrow \tilde{G}, \ i(z) = (z, 0)\).

The ambient metric \(\tilde{g}\) is a metric on \(\tilde{G}\) of signature \((p + 1, q + 1)\).
Given conformal manifold \( (M, [g]) \) of signature \( (p, q) \), \( p + q = n \).

Obtain a manifold \( \tilde{\mathcal{G}} \) of dimension \( n + 2 \) with an embedded hypersurface \( \mathcal{G} \subset \tilde{\mathcal{G}} \) and a formal expansion along \( \mathcal{G} \) for a metric \( \tilde{g} \) on \( \tilde{\mathcal{G}} \) of signature \( (p + 1, q + 1) \).

Let \( \mathcal{G} = \{(x, g_x) : x \in M, g \in [g]\} \subset S^2 T^* M \). Metric bundle.

Dilations \( \delta_s: \mathcal{G} \rightarrow \mathcal{G} \quad \delta_s(x, g_x) = (x, s^2 g_x) \)

Set \( \tilde{\mathcal{G}} = \mathcal{G} \times (-1, 1) \). Inclusion: \( \iota: \mathcal{G} \rightarrow \tilde{\mathcal{G}}, \quad \iota(z) = (z, 0) \).

The ambient metric \( \tilde{g} \) is a metric on \( \tilde{\mathcal{G}} \) of signature \( (p + 1, q + 1) \). Satisfies:
Given conformal manifold \((M, [g])\) of signature \((p, q), p + q = n\).

Obtain a manifold \(\tilde{G}\) of dimension \(n + 2\) with an embedded hypersurface \(G \subset \tilde{G}\) and a formal expansion along \(G\) for a metric \(\tilde{g}\) on \(\tilde{G}\) of signature \((p + 1, q + 1)\).

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$\mathcal{G} =$ null cone of $|x|^2 - |y|^2$
Leistner-Nurowski Result

Put these together:

\[ D \subset TM^5 \xrightarrow{\text{Nurowski}} (M, [g]) \xrightarrow{\text{Ambientmetric}} (\tilde{G}^7, \tilde{g}) \]
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Nurowski (2007). Consider $\mathcal{D} \subset T\mathbb{R}^5$ given by

$$F = q^2 + \sum_{k=0}^{6} a_k p^k + bz, \quad a_k, b \in \mathbb{R}$$
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**Theorem.** (Leistner-Nurowski, 2009) \( F \) as above.

- For all \( a_k, b \), have \( \text{Hol}(\tilde{G}, \tilde{g}) \subset G_2 \)
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- For all $a_k, b$, have $\text{Hol}(\tilde{G}, \tilde{g}) \subset G_2$
- If one of $a_3, a_4, a_5, a_6 \neq 0$, then $\text{Hol}(\tilde{G}, \tilde{g}) = G_2$. 
Leistner-Nurowski Result

Gives a completely explicit 8-parameter family of metrics of holonomy $G_2$. 
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But what about $\tilde{g}$ for other $\mathcal{D}$?
Work with Travis Willse.
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**Theorem.** Let $\mathcal{D} \subset TM^5$ be generic and real-analytic.
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The proof proceeds in 2 steps:

1. Construct $\varphi|_{\mathcal{G}}$. Should be homogeneous of degree 3.
2. Extend $\varphi|_{\mathcal{G}}$ to $\tilde{\mathcal{G}}$ to be parallel.
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Other direction is true as well:

Theorem. (Hammerl-Sagerschnig, 2009) Nurowski’s conformal structures $(M, [g])$ associated to generic $\mathcal{D}$ are characterized by the existence of a parallel tractor 3-form of split type.
Parallel Extension Theorem

Step 2. Extend $\varphi|_{\tilde{G}}$ to $\tilde{G}$ to be parallel
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We prove an ambient extension theorem for parallel tractor-tensors. Holds for general conformal structures in any signature and dimension and for parallel tractors having arbitrary symmetry. Let $\mathcal{T} =$ tractor bundle.
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We prove an ambient extension theorem for parallel tractor-tensors. Holds for general conformal structures in any signature and dimension and for parallel tractors having arbitrary symmetry.

Let $\mathcal{T}$ = tractor bundle. Tractor-tensor means a section of $\otimes^r \mathcal{T}^*$. 
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Let $\mathcal{T} =$ tractor bundle. Tractor-tensor means a section of $\otimes^r \mathcal{T}^*$. 

**Theorem.** Let $(M, [g])$ be a conformal manifold, with ambient metric $\tilde{g}$. Suppose $\varphi$ is a parallel tractor-tensor of rank $r \in \mathbb{N}$. 
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Immediately conclude $\text{Hol}(\tilde{G}, \tilde{g}) \subset G_2$. 

Conditions for $\text{Hol}(\tilde{G}, \tilde{g}) = G_2$
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Let $W_{ijkl} = \text{Weyl tensor of } g$, $C_{jkl} = \text{Cotton tensor of } g$. 
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Define $L_p : T_p M \times \mathbb{R} \to \otimes^3 T^* p M$ by

$$L(v, \lambda) = W_{ijkl} v^i + C_{jkl} \lambda.$$
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Impose: $L_p$ is injective.
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Let $A \in \Gamma(S^4 \mathcal{D}^*)$ be Cartan’s fundamental curvature invariant for generic distributions $\mathcal{D} \subset M^5$. 

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Let $A \in \Gamma(S^4\mathcal{D}^*)$ be Cartan’s fundamental curvature invariant for generic distributions $\mathcal{D} \subset M^5$. $A$ is a binary quartic.
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Let $A \in \Gamma(S^4D^*) \text{ be Cartan's fundamental curvature invariant for generic distributions } D \subset M^5$. $A$ is a binary quartic.

Say that $A$ is 3-nondegenerate at $p$ if the only vector $Y \in D_p$ such that $A(X, Y, Y, Y) = 0$ for all $X \in D_p$ is $Y = 0$. 
Conditions for $\text{Hol} (\tilde{\mathcal{G}}, \tilde{g}) = G_2$

Let $W_{ijkl} = \text{Weyl tensor of } g$, $C_{jkl} = \text{Cotton tensor of } g$.

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**Theorem.** Given $(M, \mathcal{D})$ real analytic. If there are $p, q \in M$ so that $L_p$ is injective and $A_q$ is 3-nondegenerate, then $\tilde{g}$ has holonomy $= G_2$. 
In particular, have \( \text{Hol}(\tilde{G}, \tilde{g}) = G_2 \) if there is \( p \in M \) so that \( L_p \) is injective and \( A_p \) is 3-nondegenerate.
Conditions for $\text{Hol}(\tilde{\mathcal{G}}, \tilde{g}) = G_2$

In particular, have $\text{Hol}(\tilde{\mathcal{G}}, \tilde{g}) = G_2$ if there is $p \in M$ so that $L_p$ is injective and $A_p$ is 3-nondegenerate.

Each condition is an algebraic condition on the 7-jet of $F$ at $p$. 
In particular, have $\text{Hol}(\tilde{\mathcal{G}}, \tilde{g}) = G_2$ if there is $p \in M$ so that $L_p$ is injective and $\mathcal{A}_p$ is 3-nondegenerate.

Each condition is an algebraic condition on the 7-jet of $F$ at $p$.

So if the 7-jet of $F$ avoids a particular algebraic set at a single point, then $\text{Hol}(\tilde{\mathcal{G}}, \tilde{g}) = G_2$. 
In particular, have $\Hol(\tilde{G}, \tilde{g}) = G_2$ if there is $p \in M$ so that $L_p$ is injective and $A_p$ is 3-nondegenerate.

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So if the 7-jet of $F$ avoids a particular algebraic set at a single point, then $\Hol(\tilde{G}, \tilde{g}) = G_2$.

This is a weak condition, explicitly checkable.