The Geometry of Tensor Network States

J.M. Landsberg, Yang Qi and Ke Ye

Texas A&M University

April 5, 2011

Geometry of Tensors

Many beautiful questions in geometry come from computer science, signal processing and other areas regarding tensors. This talk will focus on a question arising in quantum mechanics.

Geometry of Tensors

Many beautiful questions in geometry come from computer science, signal processing and other areas regarding tensors. This talk will focus on a question arising in quantum mechanics.

First review: What is a tensor?

Geometry of tensors: Review of linear algebra

Let V_1, V_2 be vector spaces. An element $T \in V_1 \otimes V_2$ may be viewed as a linear map $T_1 : V_1^* \to V_2$ or a linear map $T_2 : V_2^* \to V_1$. T is said to be *rank one* if there exists $v \in V_1$, $w \in V_2$ such that $T = v \otimes w$, where, e.g.,

$$\begin{aligned} v \otimes w : V_1^* &\to V_2 \\ \alpha &\mapsto \alpha(v) w \end{aligned}$$

Geometry of tensors: Review of linear algebra

Let V_1, V_2 be vector spaces. An element $T \in V_1 \otimes V_2$ may be viewed as a linear map $T_1 : V_1^* \to V_2$ or a linear map $T_2 : V_2^* \to V_1$. T is said to be *rank one* if there exists $v \in V_1$, $w \in V_2$ such that $T = v \otimes w$, where, e.g.,

$$\begin{aligned} v \otimes w : V_1^* &\to V_2 \\ \alpha &\mapsto \alpha(v) w \end{aligned}$$

In general, the *rank* of an element $T \in V \otimes W$ is the smallest r such that there exist $v_1, \ldots, v_r \in V_1$, $w_1, \ldots, w_r \in V_2$ such that $T = v_1 \otimes w_1 + \cdots + v_r \otimes w_r$.

Geometry of tensors: Review of linear algebra

Let V_1, V_2 be vector spaces. An element $T \in V_1 \otimes V_2$ may be viewed as a linear map $T_1 : V_1^* \to V_2$ or a linear map $T_2 : V_2^* \to V_1$. T is said to be *rank one* if there exists $v \in V_1$, $w \in V_2$ such that $T = v \otimes w$, where, e.g.,

$$\begin{aligned} v \otimes w : V_1^* &\to V_2 \\ \alpha &\mapsto \alpha(v) w \end{aligned}$$

In general, the *rank* of an element $T \in V \otimes W$ is the smallest r such that there exist $v_1, \ldots, v_r \in V_1$, $w_1, \ldots, w_r \in V_2$ such that $T = v_1 \otimes w_1 + \cdots + v_r \otimes w_r$.

Fundamental theorem of linear algebra:
$\operatorname{rank}(f) = \dim f(V_1^*) = \dim f(V_2^*)$

Moreover $\operatorname{rank}(T) \leq \min\{\dim V_1, \dim V_2\}$ and for generic T, equality holds.

Finally, if a sequence of linear maps T_t of rank r has a limit T_0, then $\operatorname{rank}(T_0) \leq r$.

$T \in V_1 \otimes V_2$ is completely determined (up to $GL(V_1) \times GL(V_2)$-equivalence by its rank.

Let $\hat{\sigma}_r = \hat{\sigma}_{r, V_1 \otimes V_2} \subset V_1 \otimes V_2$ denote the set of elements of rank at most r.

Geometry of Tensors cont'd

Let $V_1, \ldots, V_n$ be vector spaces, $\dim V_i = \mathbf{v}_i$. An element $T \in V_1 \otimes \cdots \otimes V_n$ may be viewed as a linear map $T_j : V_j^* \to V_1 \otimes \cdots \otimes \hat{V}_j \otimes \cdots \otimes V_n$.
T is said to be *rank one* if there exists $v_j \in V_j$, such that $T = v_1 \otimes \cdots \otimes v_n$

Geometry of Tensors cont'd

Let $V_1, \ldots, V_n$ be vector spaces, $\dim V_i = \mathbf{v}_i$. An element $T \in V_1 \otimes \cdots \otimes V_n$ may be viewed as a linear map $T_j : V_j^* \to V_1 \otimes \cdots \otimes \hat{V}_j \otimes \cdots \otimes V_n$.
T is said to be *rank one* if there exists $v_j \in V_j$, such that $T = v_1 \otimes \cdots \otimes v_n$
The *rank* of T is the smallest r such that there exist $v_{j,1}, \ldots, v_{j,r} \in V_j$, such that
$T = v_{1,1} \otimes \cdots \otimes v_{n,1} + \cdots + v_{1,r} \otimes \cdots \otimes v_{n,r}$.

The Fundamental theorem of linear algebra is false for tensors

If $T \in V_1 \otimes \cdots \otimes V_n$ is generic, then $\operatorname{rank}(T) \sim \frac{\mathbf{v}_1 \cdots \mathbf{v}_n}{\mathbf{v}_1 + \cdots + \mathbf{v}_n} >> \mathbf{v}_i$.
Rank can jump up (or down) under limits.

Geometry of Tensors: typical varieties studied

Let $\hat{\sigma}_r = \hat{\sigma}_{r,V_1\otimes\cdots\otimes V_n} \subset V_1\otimes\cdots\otimes V_n$ denote the closure of the set of elements of rank at most r. (This is called the (cone over the) r-th secant variety of the Segre variety.)

Geometry of Tensors: typical varieties studied

Let $\hat{\sigma}_r = \hat{\sigma}_{r,V_1\otimes\cdots\otimes V_n} \subset V_1\otimes\cdots\otimes V_n$ denote the closure of the set of elements of rank at most r. (This is called the (cone over the) r-th secant variety of the Segre variety.)

When $n = 2$ these are the only $GL(V_1) \times \cdots \times GL(V_n)$ varieties in the space, however when $n > 2$ there are numerous others. For example:

Geometry of Tensors: typical varieties studied

Let $\hat{\sigma}_r = \hat{\sigma}_{r,V_1\otimes\cdots\otimes V_n} \subset V_1\otimes\cdots\otimes V_n$ denote the closure of the set of elements of rank at most r. (This is called the (cone over the) r-th secant variety of the Segre variety.)

When $n = 2$ these are the only $GL(V_1)\times\cdots\times GL(V_n)$ varieties in the space, however when $n > 2$ there are numerous others. For example: Let $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1\otimes\cdots\otimes V_n)$ denote the tensors T where

$\mathrm{rank}(f_i) \leq \mathbf{f}_i$, $i \leq i \leq n$, called the *subspace variety*

Quantum Mechanics: problem

Physical states $\leftrightarrow$ Points of a Hilbert space $\mathcal{H}$

$\mathcal{H}$ has a tensor product structure: each system (e.g. molecule) is a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if there are n molecules.

Quantum Mechanics: problem

Physical states $\leftrightarrow$ Points of a Hilbert space $\mathcal{H}$

$\mathcal{H}$ has a tensor product structure: each system (e.g. molecule) is a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if there are n molecules.

Exponential growth in size of space with n means computations quickly become impossible

Quantum Mechanics: problem

Physical states $\leftrightarrow$ Points of a Hilbert space $\mathcal{H}$

$\mathcal{H}$ has a tensor product structure: each system (e.g. molecule) is a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if there are n molecules.

Exponential growth in size of space with n means computations quickly become impossible

Pure states $\sim$ rank one tensors

Quantum Mechanics: problem

Physical states $\leftrightarrow$ Points of a Hilbert space $\mathcal{H}$

$\mathcal{H}$ has a tensor product structure: each system (e.g. molecule) is a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if there are n molecules.

Exponential growth in size of space with n means computations quickly become impossible

Pure states $\sim$ rank one tensors

Set of such of size $\sim \dim \mathcal{H}_1 + \cdots + \dim \mathcal{H}_n$

Quantum Mechanics: problem

Physical states $\leftrightarrow$ Points of a Hilbert space $\mathcal{H}$

$\mathcal{H}$ has a tensor product structure: each system (e.g. molecule) is a Hilbert space $\mathcal{H}_i$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if there are n molecules.

Exponential growth in size of space with n means computations quickly become impossible

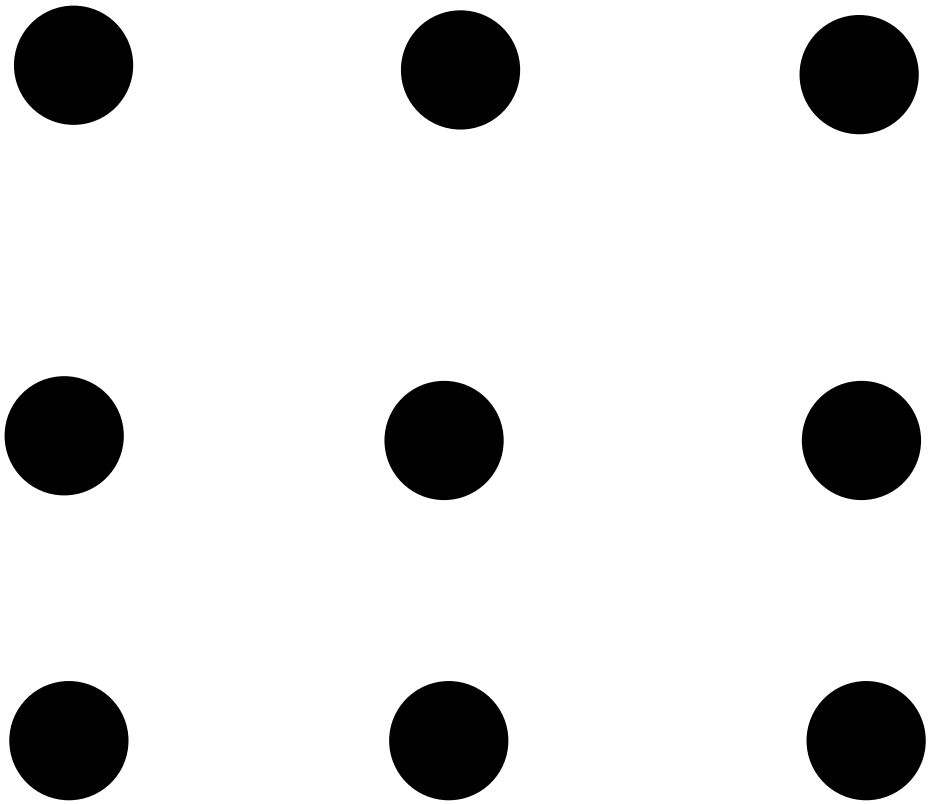
Pure states $\sim$ rank one tensors

Set of such of size $\sim \dim \mathcal{H}_1 + \cdots + \dim \mathcal{H}_n$

Entanglement only observed locally and among few bodies, usually nearby molecules - how to describe this small corner of Hilbert space?

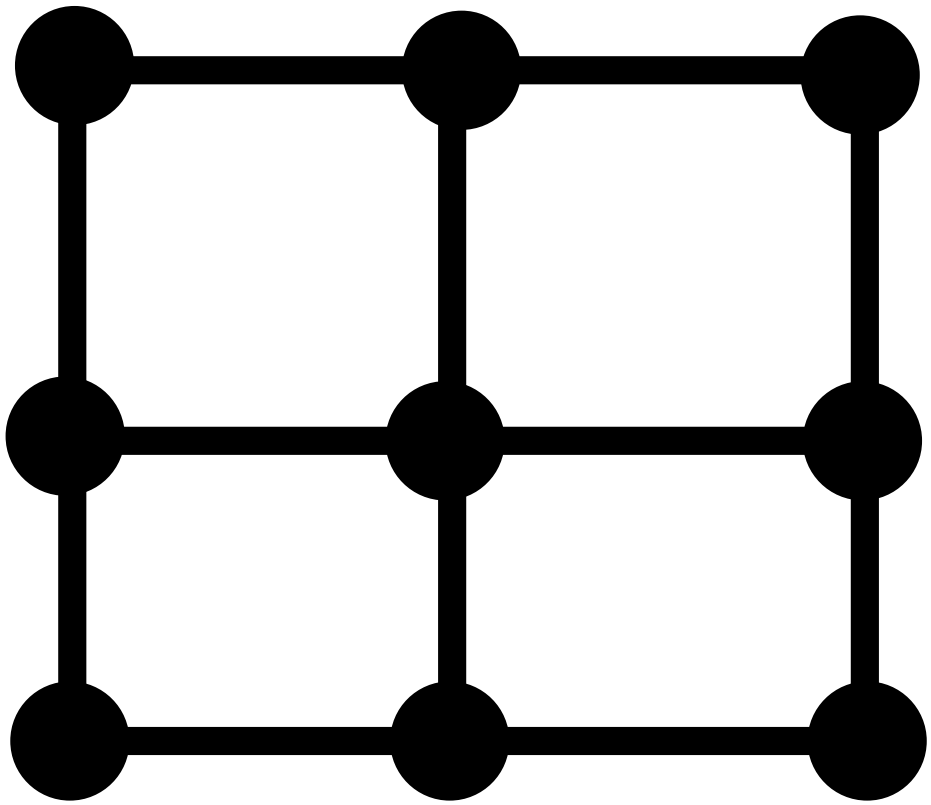
Quantum Mechanics: solution

Solution will not depend on Hilbert space structure so ignore it and just deal with vector spaces V, V_i. Given molecules arranged in a lattice



we would like to describe a set of tensors representing states where molecules are only entangled with their neighbors, and we would like to bound the amount of entanglement as well.

Draw a graph with vertices corresponding to the molecules and draw an edge between vertices if the molecules might be entangled (e.g. between neighbors).



The vector spaces corresponding to the molecules will be associated to the vertices, and to the edges we will associate an auxiliary vector space that will control the amount of entanglement between the states of the corresponding vertices.

Quantum Mechanics: solution

Notation: For a graph Γ with edges e_s and vertices v_i, $s \in e(j)$ means e_s is incident to v_j. If Γ is directed, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Quantum Mechanics: solution

Notation: For a graph Γ with edges e_s and vertices v_i, $s \in e(j)$ means e_s is incident to v_j. If Γ is directed, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Let $V_1, \ldots, V_n$ be vector spaces, let $\mathbf{v}_i = \dim V_i$. Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. Let Γ be a graph with n vertices v_j, $1 \leq j \leq n$, and m edges e_s, $1 \leq s \leq m$, and let $\vec{\mathbf{e}} = (\mathbf{e}_1, \ldots, \mathbf{e}_m) \in \mathbb{N}^m$.

Quantum Mechanics: solution

Notation: For a graph Γ with edges e_s and vertices v_i, $s \in e(j)$ means e_s is incident to v_j. If Γ is directed, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Let $V_1, \ldots, V_n$ be vector spaces, let $\mathbf{v}_i = \dim V_i$. Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. Let Γ be a graph with n vertices v_j, $1 \leq j \leq n$, and m edges e_s, $1 \leq s \leq m$, and let $\vec{\mathbf{e}} = (\mathbf{e}_1, \ldots, \mathbf{e}_m) \in \mathbb{N}^m$.

Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension $\mathbf{e}_s$ to the edge e_s. Make Γ into a directed graph. (The choice of directions will not effect the end result.)

Tensor Network States: definition

Let

$$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) := \\ \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*), \\ \text{such that } T = Con(T_1 \otimes \cdots \otimes T_n)\}$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Tensor Network States: definition

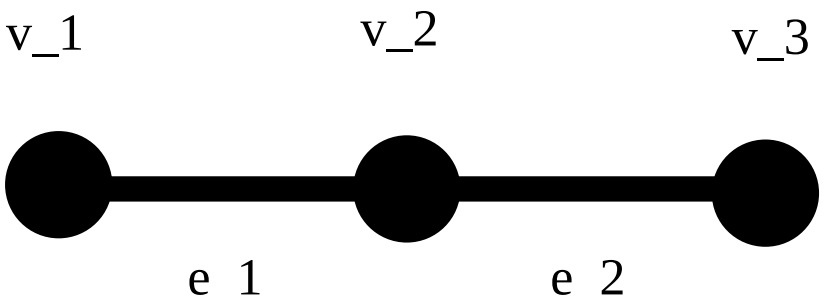
Let

$$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) := \\ \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*), \\ \text{such that } T = Con(T_1 \otimes \cdots \otimes T_n)\}$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Tensors in $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ represent states where the states of molecules are entangled with nearby molecules only. The amount of entanglement permitted is controlled by the dimensions of the auxiliary vector spaces E_s.

Example



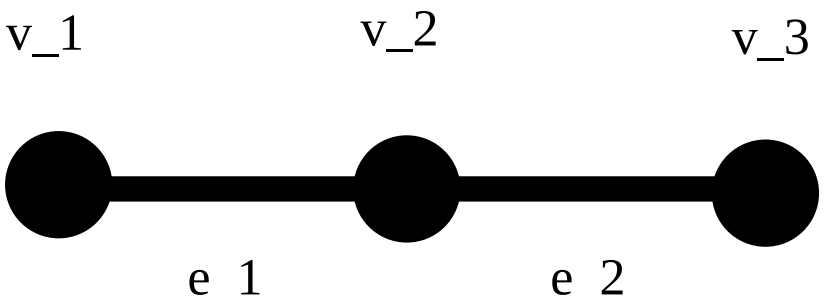
Let $\mathbf{e}_1 = \mathbf{e}_2 = 2$, let $u_j \in V_1$, $v_s \in V_2$, $w_j \in V_3$. Set

$$T_1 = e_{11} \otimes u_1 + e_{12} \otimes u_2$$

$$T_1 = e^{11} \otimes e_{21} \otimes v_1 + e^{12} \otimes e_{21} \otimes v_2 + e^{11} \otimes e_{22} \otimes v_3 + e^{12} \otimes e_{22} \otimes v_4$$

$$T_3 = e^{21} \otimes w_1 + e^{22} \otimes w_2$$

Example



Let $\mathbf{e}_1 = \mathbf{e}_2 = 2$, let $u_j \in V_1$, $v_s \in V_2$, $w_j \in V_3$. Set

$$T_1 = e_{11} \otimes u_1 + e_{12} \otimes u_2$$
$$T_1 = e^{11} \otimes e_{21} \otimes v_1 + e^{12} \otimes e_{21} \otimes v_2 + e^{11} \otimes e_{22} \otimes v_3 + e^{12} \otimes e_{22} \otimes v_4$$
$$T_3 = e^{21} \otimes w_1 + e^{22} \otimes w_2$$

Then

$$\begin{aligned} T &= Con(T_1 \otimes T_2 \otimes T_3) \\ &= u_1 \otimes v_1 \otimes w_1 + u_1 \otimes v_3 \otimes w_4 + u_2 \otimes v_2 \otimes w_1 + u_2 \otimes v_3 \otimes w_4 \end{aligned}$$

Letting $V_{ij} = V_i \otimes V_j$, we see

$$TNS(\Gamma, (2,2), V_1 \otimes V_2 \otimes V_3) = \hat{\sigma}_{2, V_1 \otimes V_{23}} \cap \hat{\sigma}_{2, V_{12} \otimes V_3}$$

is the intersection of two familiar objects.

Purpose of this talk

Give a geometric description of the sets
$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) \subset V_1 \otimes \cdots \otimes V_n$

Purpose of this talk

Give a geometric description of the sets
$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) \subset V_1 \otimes \cdots \otimes V_n$

Motivating question

(Lars Grasedyck): Given a convergent sequence of tensors $T_\epsilon \in V$, with $\lim_{\epsilon \to 0} T_\epsilon = T_0$, if $T_\epsilon \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$, must $T_0 \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$?

Purpose of this talk

Give a geometric description of the sets
$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) \subset V_1 \otimes \cdots \otimes V_n$

Motivating question

(Lars Grasedyck): Given a convergent sequence of tensors $T_\epsilon \in V$, with $\lim_{\epsilon \to 0} T_\epsilon = T_0$, if $T_\epsilon \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$, must $T_0 \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$?

I.e., is $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ (Zariski) closed?

Purpose of this talk

Give a geometric description of the sets
$TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V}) \subset V_1 \otimes \cdots \otimes V_n$

Motivating question

(Lars Grasedyck): Given a convergent sequence of tensors $T_\epsilon \in V$, with $\lim_{\epsilon \to 0} T_\epsilon = T_0$, if $T_\epsilon \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$, must $T_0 \in TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$?

I.e., is $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ (Zariski) closed?

Grasedyck mentioned he could prove it was closed if Γ was a tree, but already did not know for triangles.

Detour: Geometric Complexity Theory (GCT)

Prequel: L. Valiant proposed an algebraic analog of the **P** v. **NP** problem via affine linear projections of the determinant as follows:

Detour: Geometric Complexity Theory (GCT)

Prequel: L. Valiant proposed an algebraic analog of the **P** v. **NP** problem via affine linear projections of the determinant as follows:

Consider $\det_n \in S^n\mathbb{C}^{n^2}$, i.e. as a homogeneous polynomial of degree n in n^2 variables. Then $\mathrm{End}(\mathbb{C}^{n^2})$ acts on the space of polynomials and Valiant asked what is the largest m such that $\ell^{n-m}\,\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n$?

The conjecture is that n must grow exponentially with respect to m to have (a sequence of) inclusions.

Detour: Geometric Complexity Theory (GCT)

Prequel: L. Valiant proposed an algebraic analog of the **P** v. **NP** problem via affine linear projections of the determinant as follows:

Consider $\det_n \in S^n\mathbb{C}^{n^2}$, i.e. as a homogeneous polynomial of degree n in n^2 variables. Then $\mathrm{End}(\mathbb{C}^{n^2})$ acts on the space of polynomials and Valiant asked what is the largest m such that $\ell^{n-m}\,\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n$?

The conjecture is that n must grow exponentially with respect to m to have (a sequence of) inclusions.

Mulmuley-Sohoni variant: take closures and instead consider $\overline{GL_{n^2} \cdot \det_n} \subset S^n\mathbb{C}^{n^2}$.

Detour: Geometric Complexity Theory (GCT)

Prequel: L. Valiant proposed an algebraic analog of the **P** v. **NP** problem via affine linear projections of the determinant as follows:

Consider $\det_n \in S^n\mathbb{C}^{n^2}$, i.e. as a homogeneous polynomial of degree n in n^2 variables. Then $\mathrm{End}(\mathbb{C}^{n^2})$ acts on the space of polynomials and Valiant asked what is the largest m such that $\ell^{n-m}\,\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2}) \cdot \det_n$?

The conjecture is that n must grow exponentially with respect to m to have (a sequence of) inclusions.

Mulmuley-Sohoni variant: take closures and instead consider $\overline{GL_{n^2} \cdot \det_n} \subset S^n\mathbb{C}^{n^2}$.

Remark

Up until recently [2010, L-Manivel-Ressayre, Hypersurfaces with degenerate duals and the GCT program, to appear in CMH] it was not known if there was anything new in the closure.

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first:

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first: Consider

$$MMult = MMult_{p,q,r} : (\mathbb{C}^p \otimes \mathbb{C}^{q*}) \times (\mathbb{C}^q \otimes \mathbb{C}^{r*}) \to (\mathbb{C}^p \otimes \mathbb{C}^{r*})$$

of $p \times q$ matrices times $q \times r$ matrices to $p \times r$ matrices. Write $V_1 = \mathbb{C}^{p*} \otimes \mathbb{C}^{q*}$, $V_2 = \mathbb{C}^{q*} \otimes \mathbb{C}^r$, $V_3 = \mathbb{C}^p \otimes \mathbb{C}^{r*}$. Then $MMult \in V_1 \otimes V_2 \otimes V_3$.
Consider

$$\overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult} \subset V_1 \otimes V_2 \otimes V_3.$$

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first: Consider

$$MMult = MMult_{p,q,r} : (\mathbb{C}^p \otimes \mathbb{C}^{q*}) \times (\mathbb{C}^q \otimes \mathbb{C}^{r*}) \to (\mathbb{C}^p \otimes \mathbb{C}^{r*})$$

of $p \times q$ matrices times $q \times r$ matrices to $p \times r$ matrices. Write $V_1 = \mathbb{C}^{p*} \otimes \mathbb{C}^{q*}$, $V_2 = \mathbb{C}^{q*} \otimes \mathbb{C}^r$, $V_3 = \mathbb{C}^p \otimes \mathbb{C}^{r*}$. Then $MMult \in V_1 \otimes V_2 \otimes V_3$.
Consider

$$\overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult} \subset V_1 \otimes V_2 \otimes V_3.$$

What is its boundary?

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first: Consider

$$MMult = MMult_{p,q,r} : (\mathbb{C}^p \otimes \mathbb{C}^{q*}) \times (\mathbb{C}^q \otimes \mathbb{C}^{r*}) \to (\mathbb{C}^p \otimes \mathbb{C}^{r*})$$

of $p \times q$ matrices times $q \times r$ matrices to $p \times r$ matrices. Write $V_1 = \mathbb{C}^{p*} \otimes \mathbb{C}^{q*}$, $V_2 = \mathbb{C}^{q*} \otimes \mathbb{C}^r$, $V_3 = \mathbb{C}^p \otimes \mathbb{C}^{r*}$. Then $MMult \in V_1 \otimes V_2 \otimes V_3$.
Consider

$$\overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult} \subset V_1 \otimes V_2 \otimes V_3.$$

What is its boundary?

Note that linear projections are certainly in the boundary, i.e.,

$$[End(V_1) \times End(V_2) \times End(V_3)] \cdot MMult \subseteq \overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult}$$

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first: Consider

$$MMult = MMult_{p,q,r} : (\mathbb{C}^p \otimes \mathbb{C}^{q*}) \times (\mathbb{C}^q \otimes \mathbb{C}^{r*}) \to (\mathbb{C}^p \otimes \mathbb{C}^{r*})$$

of $p \times q$ matrices times $q \times r$ matrices to $p \times r$ matrices. Write $V_1 = \mathbb{C}^{p*} \otimes \mathbb{C}^{q*}$, $V_2 = \mathbb{C}^{q*} \otimes \mathbb{C}^r$, $V_3 = \mathbb{C}^p \otimes \mathbb{C}^{r*}$. Then $MMult \in V_1 \otimes V_2 \otimes V_3$.
Consider

$$\overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult} \subset V_1 \otimes V_2 \otimes V_3.$$

What is its boundary?

Note that linear projections are certainly in the boundary, i.e.,

$$[End(V_1) \times End(V_2) \times End(V_3)] \cdot MMult \subseteq \overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult}$$

Is there anything else?

Geometric Complexity Theory cont'd

[Bürgisser and Ikenmeyer: Geometric Complexity Theory and Tensor Rank, arXiv 2010] : since GCT program is expected to be very difficult, one should analyze the "toy" problem of matrix multiplication first: Consider

$$MMult = MMult_{p,q,r} : (\mathbb{C}^p \otimes \mathbb{C}^{q*}) \times (\mathbb{C}^q \otimes \mathbb{C}^{r*}) \to (\mathbb{C}^p \otimes \mathbb{C}^{r*})$$

of $p \times q$ matrices times $q \times r$ matrices to $p \times r$ matrices. Write $V_1 = \mathbb{C}^{p*} \otimes \mathbb{C}^{q*}$, $V_2 = \mathbb{C}^{q*} \otimes \mathbb{C}^r$, $V_3 = \mathbb{C}^p \otimes \mathbb{C}^{r*}$. Then $MMult \in V_1 \otimes V_2 \otimes V_3$.
Consider

$$\overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult} \subset V_1 \otimes V_2 \otimes V_3.$$

What is its boundary?

Note that linear projections are certainly in the boundary, i.e.,

$$[End(V_1) \times End(V_2) \times End(V_3)] \cdot MMult \subseteq \overline{(GL(V_1) \times GL(V_2) \times GL(V_3)) \cdot MMult}$$

Is there anything else?

Tensor Network States and toy GCT

Proposition

Let $\mathbf{v}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{v}_2 = \mathbf{e}_3\mathbf{e}_1, \mathbf{v}_3 = \mathbf{e}_2\mathbf{e}_1$. Then $TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ consists of matrix multiplication (up to relabeling) and its degenerations, i.e.

$$TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3) = \\ [End(V_1) \times End(V_2) \times End(V_3)] \cdot MMult_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}.$$

It has dimension $\mathbf{e}_2^2\mathbf{e}_3^2 + \mathbf{e}_2^2\mathbf{e}_1^2 + \mathbf{e}_3^2\mathbf{e}_1^2 - (\mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_1^2)$.

So the toy GCT question regarding the boundary is essentially the same as Grasedyk's question!

Tensor Network States and toy GCT

Proposition

Let $\mathbf{v}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{v}_2 = \mathbf{e}_3\mathbf{e}_1, \mathbf{v}_3 = \mathbf{e}_2\mathbf{e}_1$. Then $TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ consists of matrix multiplication (up to relabeling) and its degenerations, i.e.

$$TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3) = \\ [End(V_1) \times End(V_2) \times End(V_3)] \cdot MMult_{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1}.$$

It has dimension $\mathbf{e}_2^2\mathbf{e}_3^2 + \mathbf{e}_2^2\mathbf{e}_1^2 + \mathbf{e}_3^2\mathbf{e}_1^2 - (\mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_1^2)$.

So the toy GCT question regarding the boundary is essentially the same as Grasedyk's question!

Theorem

Let $\mathbf{v}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{v}_2 = \mathbf{e}_3\mathbf{e}_1, \mathbf{v}_3 = \mathbf{e}_2\mathbf{e}_1$. Then $TNS(\triangle, (\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_2\mathbf{e}_1), V_1 \otimes V_2 \otimes V_3)$ is not Zariski closed.

Proof

Recall matrix multiplication may be considered as a trilinear map

$$\begin{aligned} MMult : V_1^* \otimes V_2^* \otimes V_3^* &\to \mathbb{C} \\ (P, Q, R) &\mapsto \mathrm{trace}(PQR) \end{aligned}$$

We'll construct curves $g_j(t) \subset GL(V_j)$ such that for $t \neq 0$,

$$(g_1(t), g_2(t), g_3(t)) \cdot MMult \in GL(V_1) \times GL(V_2) \times GL(V_3) \cdot MMult,$$

but in the limit as $t \to 0$, one obtains a point on the boundary not in $End(V_1) \times End(V_2) \times End(V_3) \cdot MMult$.

Proof cont'd

Each $g_j(t)$ will be of the form $g_{j0} + tId$ where g_{j0} is a projection operator. Let g_{10}, g_{20} be projection operators (projecting into blocked submatrices)

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

and g_{30} of the form

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \to \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

Call the projected images P_0, Q_0, R_0 respectively and write P_1, Q_1, R_1 for their complements. The limiting operator (up to scale) is

$$(P, Q, R) \mapsto \mathrm{trace}(P_0 Q_0 R_1) + \mathrm{trace}(P_0 Q_1 R_0) + \mathrm{trace}(P_1 Q_0 R_0)$$

End of proof

One finishes the proof by verifying that the limiting map is not in the orbit, which is done by computing that its stabilizer is larger than the stabilizer of matrix multiplication, and to verify that it is not in components of the boundary obtained from $End(V_1) \times End(V_2) \times End(V_3) \cdot MMult$ by showing that it uses all the variables, i.e., that it is not in a subspace variety.

End of proof

One finishes the proof by verifying that the limiting map is not in the orbit, which is done by computing that its stabilizer is larger than the stabilizer of matrix multiplication, and to verify that it is not in components of the boundary obtained from $End(V_1) \times End(V_2) \times End(V_3) \cdot MMult$ by showing that it uses all the variables, i.e., that it is not in a subspace variety.

Remark

The most interesting limiting points have yet to be discovered as our examples have a stabilizer much larger than that of matrix multiplication, whereas all components of the boundary have codimension one, so there exist limit points with stabilizer of dimension just one greater than that of matrix multiplication.

Reduction theorems

Definition

A dimension $\mathbf{v}_i$ is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_i = \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_i \leq \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\geq$. If $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical for all i, we say $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical.

Reduction theorems

Definition

A dimension $\mathbf{v}_i$ is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_i = \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_i \leq \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\geq$. If $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical for all i, we say $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical.

Subcritical vertex Proposition

Cases with subcritical vertices are linear projections of critical cases.

Reduction theorems

Definition

A dimension $\mathbf{v}_i$ is *critical*, resp. *subcritical*, resp. *supercritical*, if $\mathbf{v}_i = \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\mathbf{v}_i \leq \Pi_{s \in e(j)} \mathbf{e}_s$, resp. $\geq$. If $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical for all i, we say $TNS(\Gamma, \vec{\mathbf{e}}, \mathbf{V})$ is critical.

Subcritical vertex Proposition

Cases with subcritical vertices are linear projections of critical cases.

Supercritical vertex Proposition

Tensor network states with supercritical vertices are towers of bundles over critical tensor network states

The supercritical reduction is more interesting, so I describe it in more detail.

Supercritical Reduction: Detour - Kempf-Weyman desingularizations

Let $\mathbf{f}_j \leq \mathbf{v}_j$ be natural numbers. Recall the subspace variety:

$$Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n) := \\ \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists F_j \subset V_j,\ \dim F_j = \mathbf{f}_j, \\ \text{such that } T \in F_1 \otimes \cdots \otimes F_n\}.$$

Supercritical Reduction: Detour - Kempf-Weyman desingularizations

Let $\mathbf{f}_j \leq \mathbf{v}_j$ be natural numbers. Recall the subspace variety:

$$\begin{aligned} &Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n) := \\ &\quad \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists F_j \subset V_j,\ \dim F_j = \mathbf{f}_j, \\ &\quad \text{such that } T \in F_1 \otimes \cdots \otimes F_n\}. \end{aligned}$$

While $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n)$ is not homogeneous, it is nearly so.

Supercritical Reduction: Detour - Kempf-Weyman desingularizations

Let $\mathbf{f}_j \leq \mathbf{v}_j$ be natural numbers. Recall the subspace variety:

$$\begin{aligned} &Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n) := \\ &\quad \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists F_j \subset V_j,\ \dim F_j = \mathbf{f}_j, \\ &\quad \text{such that } T \in F_1 \otimes \cdots \otimes F_n\}. \end{aligned}$$

While $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n)$ is not homogeneous, it is nearly so.

Kempf and Weyman developed methods to study varieties such as $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n)$ by what Kempf called the *collapsing of a homogeneous vector bundle*.

Supercritical Reduction - Kempf-Weyman desingularizations cont'd

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E.

Supercritical Reduction - Kempf-Weyman desingularizations cont'd

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E.
Consider the bundle $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \to G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$.
It is a subbundle of the trivial bundle with fiber $V_1 \otimes \cdots \otimes V_n$ and thus its total space has a projection to $V_1 \otimes \cdots \otimes V_n$. The image is the subspace variety $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n)$.

Supercritical Reduction - Kempf-Weyman desingularizations cont'd

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E.
Consider the bundle $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \to G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$.
It is a subbundle of the trivial bundle with fiber $V_1 \otimes \cdots \otimes V_n$ and thus its total space has a projection to $V_1 \otimes \cdots \otimes V_n$. The image is the subspace variety $Sub_{\mathbf{f}_1,\ldots,\mathbf{f}_n}(V_1 \otimes \cdots \otimes V_n)$.
Essentially reduces the study of the subspace variety to a study of homogeneous bundles.

Supercritical Reduction

The supercritical cases may be realized as towers of bundles over the critical cases as follows:

Supercritical Proposition - precise version

Assume $\mathbf{f}_j := \Pi_{s \in e(j)} \mathbf{e}_s \leq \mathbf{v}_j$. Then $TNS(\Gamma, \vec{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n)$ may be described as the collapsing of a fiber bundle $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$ over $G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ where $\mathcal{S}_j \to G(\mathbf{f}_j, V_j)$ are the tautological subspace bundles. The fiber over $([F_1], \ldots, [F_n]) \in G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ is just $TNS(\Gamma, \vec{\mathbf{e}}, F_1 \otimes \cdots \otimes F_n)$

Supercritical Reduction

The supercritical cases may be realized as towers of bundles over the critical cases as follows:

Supercritical Proposition - precise version

Assume $\mathbf{f}_j := \Pi_{s\in e(j)}\mathbf{e}_s \leq \mathbf{v}_j$. Then $TNS(\Gamma, \vec{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n)$ may be described as the collapsing of a fiber bundle $TNS(\Gamma, \vec{\mathbf{e}}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$ over $G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ where $\mathcal{S}_j \to G(\mathbf{f}_j, V_j)$ are the tautological subspace bundles. The fiber over $([F_1], \ldots, [F_n]) \in G(\mathbf{f}_1, V_1) \times \cdots \times G(\mathbf{f}_n, V_n)$ is just $TNS(\Gamma, \vec{\mathbf{e}}, F_1 \otimes \cdots \otimes F_n)$
In particular

$$\dim(TNS(\Gamma, \vec{\mathbf{e}}, V_1 \otimes \cdots \otimes V_n))$$
$$= \dim(TNS(\Gamma, \vec{\mathbf{e}}, \mathbb{C}^{\mathbf{f}_1} \otimes \cdots \otimes \mathbb{C}^{\mathbf{f}_n})) + \sum_{j=1}^{n} \mathbf{f}_j(\mathbf{v}_j - \mathbf{f}_j).$$

Thank you