Canonical directions on surfaces in $M^2(c) \times \mathbb{R}$

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Canonical coordinates and principal directions

1. **The ambient space** $M^2(c) \times \mathbb{R}$
   - Constant Angle Surfaces in $M^2(c) \times \mathbb{R}$

2. **Surfaces in** $S^2 \times \mathbb{R}$

3. **Surfaces in** $H^2 \times \mathbb{R}$
   - Minkowski model of $H^2$
   - Minimality and Flatness

4. **Surfaces in Euclidean space** $\mathbb{E}^3$
The ambient space $\mathbb{M}^2(c) \times \mathbb{R}$

- $c = 1 \Rightarrow \mathbb{M}^2(c) = \mathbb{S}^2 \Rightarrow$ the ambient space $\mathbb{S}^2 \times \mathbb{R}$
- $c = -1 \Rightarrow \mathbb{M}^2(c) = \mathbb{H}^2 \Rightarrow$ the ambient space $\mathbb{H}^2 \times \mathbb{R}$
- $c = 0 \Rightarrow \mathbb{M}^2(c) = \mathbb{R}^2 \Rightarrow$ the ambient space $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$


Problem 1: Constant Angle Surfaces

A problem studied until now consists of the classification and characterization of Constant Angle Surfaces (CAS) in different ambient spaces. A CAS is an orientable surface whose unit normal makes a constant angle, denoted by $\theta$, with a fixed direction.

The complete classification:


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The complete classification:


Eugenio Garnica, Oscar Palmas, Gabriel Ruiz-Hernandez, Hypersurfaces making a constant angle with a closed conformal vector.
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Problem 2: Canonical directions

When the ambient is of the form $M^2 \times \mathbb{R}$, a favored direction is $\mathbb{R}$. It is known that for a constant angle surface in $E^3$, $S^2 \times \mathbb{R}$ or in $H^2 \times \mathbb{R}$, the projection of $\frac{\partial}{\partial t}$ (where $t$ is the global parameter on $\mathbb{R}$) onto the tangent plane of the immersed surface, denoted by $T$, is a principal direction with the corresponding principal curvature identically zero.

Question

Study surfaces in $M^2 \times \mathbb{R}$ such that $T$ remains a principal direction but with the corresponding principal curvature different from 0.
Problem 2: Canonical directions

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First answer in $S^2 \times \mathbb{R}$

The characterization of surfaces with a principal direction:

**Theorem (Dillen, Fastenakels, Van der Veken, 2009)**

Let $M$ be an immersed surface in $S^2 \times \mathbb{R}$ and $p$ a point of $M$ for which $\theta(p) \neq \{0, \frac{\pi}{2}\}$. Then $T$ is a principal direction if and only if $M$ considered as a surface in $\mathbb{E}^4$ is normally flat.
First answer in $S^2 \times \mathbb{R}$

Proposition (classification result) - Dillen, Fastenakels, Van der Veken, 2009

A surface $M$ immersed in $S^2 \times \mathbb{R}$ is a surface for which $T$ is a principal direction if and only if the immersion $F$ is (up to isometries of $S^2 \times \mathbb{R}$) in the neighborhood of a point $p$ where $\theta(p) \notin \{0, \frac{\pi}{2}\}$ given by

$$F : M \to S^2 \times \mathbb{R} : (x, y) \mapsto (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x))$$

with

$$F_j(x, y) = \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv$$

for $j = 1, 2, 3$ where $\phi'(x) = \cos(\theta(x))$, $F_4'(x) = \sin(\theta(x))$, $(\alpha_1, \alpha_2, \alpha_3)$ is a curve in $S^2$ and $F_1^2 + F_2^2 + F_3^2 = 1$. Moreover, $\alpha_1, \alpha_2, \alpha_3, \psi$ and $\phi$ are related by

$$\alpha'_j(y) = -\cos(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \cos(\psi(x) + \phi(v)) dv$$

and

$$-\sin(\psi(x) + \phi(y)) \int_{y_0}^{y} \alpha_j(v) \sin(\psi(x) + \phi(v)) dv.$$
General things in $H^2 \times \mathbb{R}$

Notations:
- $\tilde{M} = H^2 \times \mathbb{R}$ the Riemannian product of $(H^2(-1), g_H)$ and $\mathbb{R}$
- $\tilde{g} = g_H + dt^2$ the product metric, $t$ the (global) coordinate on $\mathbb{R}$
- $\tilde{\nabla}$ the Levi Civita connection of $\tilde{g}$
- $\partial_t := \frac{\partial}{\partial t}$ the unit vector field tangent to the $\mathbb{R}$-direction
- $\tilde{R}$ either the curvature tensor $\tilde{R}(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$, or the Riemann-Christoffel tensor on $\tilde{M}$ defined by $\tilde{R}(W, Z, X, Y) = \tilde{g}(W, \tilde{R}(X, Y)Z)$.
- $F : M \longrightarrow \tilde{M}$ - isometric immersion (dim $M = 2$)
- $\xi$ - a unit normal vector to $M$, $A$ - its shape operator
- $g = \tilde{g}|_M$ - metric on $M$, $\nabla$ - corresponding Levi Civita connection

\[(G) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad h \text{ the second fundamental form of } M \]
\[(W) \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \]

Marian Ioan Munteanu (UAIC)
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Some useful formulas

Since \( \partial_t := \frac{\partial}{\partial t} \) is of unit length, we decompose it as
\[
\partial_t = T + \cos \theta \, \xi
\]
where
- \( T \) is the projection on \( T(M) \) with \( |T| = \sin \theta \)
- \( \theta \) is the angle function : \( \cos \theta = \tilde{g}(\partial_t, \xi) \).

(E.G.)
\[
R(X, Y, Z, W) = g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) - g(X, W)g(Y, Z) + g(X, Z)g(Y, W) + g(X, W)g(Y, T)g(Z, T) + g(Y, Z)g(X, T)g(W, T) - g(X, Z)g(Y, T)g(W, T) - g(Y, W)g(X, T)g(Z, T)
\]

(E.C.)
\[
(\nabla_X A) Y - (\nabla_Y A) X = \cos \theta \, (g(X, T)Y - g(Y, T)X)
\]

Computing the Gaussian curvature \( K \), from the equation of Gauss it follows

\[
K = \det A - \cos^2 \theta.
\]

Any vector field \( X \in T(M) \) can be decomposed as

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X = X_H + g(X, T)\partial_t
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\textbf{(E.G.)} $\quad R(X, Y, Z, W) = g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) - $ $g(X, W)g(Y, Z) + g(X, Z)g(Y, W) + $ $g(X, W)g(Y, T)g(Z, T) + g(Y, Z)g(X, T)g(W, T) - $ $g(X, Z)g(Y, T)g(W, T) - g(Y, W)g(X, T)g(Z, T)$

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Any vector field $X \in T(M)$ can be decomposed as $X = X_H + g(X, T)\partial_t$.
Proposition (Dillen, M., 2009)

For an arbitrary vector $X$ tangent to $M$ we have

$$\nabla_X T = \cos \theta AX$$  \hspace{1cm} (1)

$$X(\cos \theta) = -g(AX, T).$$  \hspace{1cm} (2)

For $\theta = \text{const.}$, eq. (2) yields $g(AT, X) = 0$, $\forall X \in T(M)$. Hence:

- if $T = 0$ on $M$, then $\partial_t$ is always normal, so $M \subseteq H^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$.
- if $T \neq 0$ then $T$ is a principal direction with principal curvature 0.

Question

Study surfaces in $H^2 \times \mathbb{R}$ such that $T$ remains a principal direction but with the corresponding principal curvature different from 0.
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Question

**Study surfaces in $\mathbb{H}^2 \times \mathbb{R}$ such that $T$ remains a principal direction but with the corresponding principal curvature different from $0$.**
First answers

Further on we suppose that $\theta$ is different from 0 and $\frac{\pi}{2}$.

Proposition (Dillen, M., Nistor, to appear Taiwanese J. Math., 2011)

If $\theta \neq 0, \frac{\pi}{2}$, then we can choose local coordinates $(x, y)$ on the surface $M$ isometrically immersed in $\tilde{M}$ with $\partial_x$ in the direction of $T$ such that

$$ g(x, y) = \frac{1}{\sin^2 \theta} dx^2 + \beta^2(x, y) dy^2 \quad (3) $$

$$ A = \left( \begin{array}{cc} \theta_x \sin \theta & \theta_y \sin \theta \\ \theta_y & \sin^2 \theta \beta_x \\ \sin \theta \beta^2 & \cos \theta \beta \end{array} \right) \quad (4) $$

and the functions $\theta$ and $\beta$ are related by the PDE

$$ \frac{\sin^2 \theta}{\cos \theta} \frac{\beta_{xx}}{\beta} + \frac{\sin \theta \theta_x}{\cos^2 \theta} \frac{\beta_x}{\beta} + \frac{\theta_y}{\sin \theta} \frac{\beta_y}{\beta^3} + \left( 2 \frac{\cos \theta \theta_y^2}{\sin^2 \theta} - \frac{\theta_{yy}}{\sin \theta} \right) \frac{1}{\beta^2} - \cos \theta = 0. \quad (5) $$
An analogue result formulated for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ having $T$ as principal direction, is the following

**Proposition (Dillen, M., Nistor, 2011)**

Let $M$ be isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ with $T$ a principal direction. Then, we can choose the local coordinates $(x, y)$ such that $\partial_x$ is in the direction of $T$,

$$g = dx^2 + \beta^2(x, y)dy^2$$

$$A = \begin{pmatrix} \theta_x & 0 \\ 0 & \tan \theta \frac{\beta_x}{\beta} \end{pmatrix}.$$  

Moreover, the functions $\theta$ and $\beta$ are related by the PDE

$$\beta_{xx} + \tan \theta \theta_x \beta_x - \beta \cos^2 \theta = 0$$

and $\theta_y = 0$. 
Canonical coordinates

Remark

For every two functions $\theta$ and $\beta$ defined on a smooth simply connected surface $M$ such that $\theta_y = 0$ and $\beta_{xx} + \tan \theta \theta_x \beta_x - \beta \cos^2 \theta = 0$ for certain coordinates $(x, y)$, we can construct an isometric immersion $F : M \to \mathbb{H}^2 \times \mathbb{R}$ with the shape operator (7) and such that it has a canonical principal direction.

Remark

Let $M$ be an isometrically immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ such that $T$ is a principal direction. The coordinates $(x, y)$ on $M$ such that $\partial_x$ is collinear with $T$ and the metric $g$ has the form $g = dx^2 + \beta^2(x, y)dy^2$ are called canonical coordinates. Of course, they are not unique. More precisely, if $(x, y)$ and $(\bar{x}, \bar{y})$ are both canonical coordinates, then they are related by $\bar{x} = \pm x + c$ and $\bar{y} = \bar{y}(y)$, where $c$ is a real constant.
**Minkowski model of the hyperbolic plane** \( \mathbb{H}^2 \)

Models for the hyperbolic plane:

1. the Klein model
2. the Poincaré disk
3. the upper half plane \( \mathbb{H}^+ \)
4. Minkowski model \( \mathcal{H} \)

\[
\mathbb{H}^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid x_1^2 + x_2^2 - x_3^2 = -1, \ x_3 > 0 \right\}
\]

with Lorentzian metric

\[
\langle \ , \ \rangle = dx_1^2 + dx_2^2 - dx_3^2
\]

having constant Gaussian curvature \(-1\).
Minkowski model of the hyperbolic plane $H^2$

Models for the hyperbolic plane:

1. the Klein model
2. the Poincaré disk
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$$H^2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid x_1^2 + x_2^2 - x_3^2 = -1, \ x_3 > 0 \}$$

with Lorentzian metric

$$\langle \ , \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

having constant Gaussian curvature $-1$. 
Characterization theorem

In order to study under which conditions $T$ is a canonical principal direction, we regard the surface $M$ as a surface immersed in $\mathbb{R}^3_1 \times \mathbb{R}$ (also denoted $\mathbb{R}^4_1$) having codimension 2. The metric on the ambient space is given by $\tilde{g} = dx_1^2 + dx_2^2 - dx_3^2 + dt^2$. $M$ is given by the immersion $F : M \to \mathbb{R}^3_1 \times \mathbb{R}$, $F = (F_1, F_2, F_3, F_4)$.

**Theorem (Dillen, M., Nistor, 2011)**

Let $M$ be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$. Then $T$ is a principal direction if and only if $M$ is normally flat in $\mathbb{R}^4_1$. 
Classification theorem - version 1

**Theorem (Dillen, M., Nistor, 2011)**

If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \neq 0, \frac{\pi}{2}$, then $T$ is a principal direction if and only if $F$ is given, up to isometries of $\mathbb{H}^2 \times \mathbb{R}$, by

$$F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x))$$

with

$$F_j(x, y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x), \quad j = 1, 3$$

and

$$F_4(x) = \int_0^x \sin \theta(\tau)d\tau, \text{ where } \phi'(x) = \cos \theta.$$ The six functions $A_j$ and $B_j$ are found in one of the following cases

- **Case 1.**

  $$A_j(y) = \int_0^y H_j(\tau) \cosh \psi(\tau)d\tau + c_{1j}$$

  $$B_j(y) = \int_0^y H_j(\tau) \sinh \psi(\tau)d\tau + c_{2j}$$

  $$H_j'(y) = B_j(y) \sinh \psi(y) - A_j(y) \cosh \psi(y)$$
Classification theorem - version 1

Theorem (Dillen, M., Nistor, 2011)

If \( F : M \to \mathbb{H}^2 \times \mathbb{R} \) is an isometric immersion with \( \theta \neq 0, \frac{\pi}{2} \), then \( T \) is a principal direction if and only if \( F \) is given, up to isometries of \( \mathbb{H}^2 \times \mathbb{R} \), by

\[
F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x))
\]

with \( F_j(x, y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x) \), \( j = 1, 3 \) and

\[
F_4(x) = \int_0^x \sin \theta(\tau)d\tau, \text{ where } \phi'(x) = \cos \theta.
\]

The six functions \( A_j \) and \( B_j \) are found in one of the following cases

- **Case 2.**

\[
A_j(y) = \int_0^y H_j(\tau) \sinh \psi(\tau)d\tau + c_{1j}
\]

\[
B_j(y) = \int_0^y H_j(\tau) \cosh \psi(\tau)d\tau + c_{2j}
\]

\[
H_j'(y) = -A_j(y) \sinh \psi(y) + B_j(y) \cosh \psi(y)
\]
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Theorem (Dillen, M., Nistor, 2011)

If \( F : M \rightarrow H^2 \times \mathbb{R} \) is an isometric immersion with \( \theta \neq 0, \frac{\pi}{2} \), then \( T \) is a principal direction if and only if \( F \) is given, up to isometries of \( H^2 \times \mathbb{R} \), by

\[
F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x))
\]

with \( F_j(x, y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x), \ j = 1, 3 \) and

\[
F_4(x) = \int_0^x \sin \theta(\tau) d\tau, \text{ where } \phi'(x) = \cos \theta. \text{ The six functions } A_j \text{ and } B_j \text{ are found in one of the following cases}
\]

- **Case 3.**

\[
A_j(y) = \pm \int_0^y H_j(\tau) d\tau + c_{1j}
\]

\[
B_j(y) = \int_0^y H_j(\tau) d\tau + c_{2j}
\]

\[
H_j'(y) = c_{2j} \mp c_{1j}
\]
Classification theorem - version 1

Theorem (Dillen, M., Nistor, 2011)

If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \neq 0, \frac{\pi}{2}$, then $T$ is a principal direction if and only if $F$ is given, up to isometries of $\mathbb{H}^2 \times \mathbb{R}$, by

$$F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x))$$

with $F_j(x, y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x)$, $j = 1, 3$ and $F_4(x) = \int_0^x \sin \theta(\tau) d\tau$, where $\phi'(x) = \cos \theta$. The six functions $A_j$ and $B_j$ are found in one of the following cases

where $H = (H_1, H_2, H_3)$ is a curve on the de Sitter space $\mathbb{S}^2_1$, $\psi$ is a smooth function on $M$ and $c_1 = (c_{11}, c_{12}, c_{13})$, $c_2 = (c_{21}, c_{22}, c_{23})$ are constant vectors.
Classification theorem - version 2

Theorem (Dillen, M., Nistor, 2011)

If \( F: M \rightarrow \mathbb{H}^2 \times \mathbb{R} \) is an isometric immersion with angle function \( \theta \neq 0, \frac{\pi}{2} \), then \( T \) is a principal direction if and only if \( F \) is given locally, up to isometries of the ambient space by

\[
F(x, y) = (A(y) \sinh \phi(x) + B(y) \cosh \phi(x), \chi(x))
\]

where \( A(y) \) is a regular curve in \( S^2_1 \), \( B(y) \) is a regular curve in \( \mathbb{H}^2 \), such that \( \langle A, B \rangle = 0 \), \( A' \parallel B' \) and where \( (\phi(x), \chi(x)) \) is a regular curve in \( \mathbb{R}^2 \). The angle function \( \theta \) of \( M \) depends only on \( x \) and it coincides with the angle function of the curve \( (\phi, \chi) \). In particular, we may arc length reparametrize \( (\phi, \chi) \); then \( (x, y) \) are canonical coordinates and \( \theta'(x) = \kappa(x) \), the curvature of \( (\phi, \chi) \).
Clasiffication theorem - version 3

Theorem (Dillen, M., Nistor, 2011)

Let \( F : M \to \mathbb{H}^2 \times \mathbb{R} \) be an isometrically immersed surface \( M \) in \( \mathbb{H}^2 \times \mathbb{R} \), with \( \theta \neq 0, \pi/2 \). Then \( M \) has \( T \) as a principal direction if and only if \( F \) is given, up to rigid motions of the ambient space, by

\[
F(x, y) = \left( f(y) \cosh \phi(x) + N_f(y) \sinh \phi(x), \chi(x) \right)
\]

where \( f(y) \) is a regular curve in \( \mathbb{H}^2 \) and \( N_f(y) = \frac{f(y) \otimes f'(y)}{\sqrt{\langle f'(y), f'(y) \rangle}} \) represents the normal of \( f \). Moreover, \( (\phi, \chi) \) is a regular curve in \( \mathbb{R}^2 \) and the angle function \( \theta \) of this curve is the same as the angle function of the surface parameterized by \( F \).
Examples

Now, we would like to give some examples of surfaces that can be retrieved from the classification theorem. Let us consider first $\psi(y) = 0$ for all $y$ in Case 1, getting

$$A_j(y) = \int_0^y H_j(\tau) d\tau + c_{1j}, \quad B_j(y) = c_{2j}, \quad H'_j(y) = -\int_0^y H_j(\tau) d\tau - c_{1j}.$$ 

The parametrization $F$ in this case is given by

**Example (rotational surface)**

$$F(x, y) = \left( \sin y \sinh \left( \int_0^x \cos \theta(\tau) d\tau \right), \cos y \sinh \left( \int_0^x \cos \theta(\tau) d\tau \right), \cosh \left( \int_0^x \cos \theta(\tau) d\tau \right), \int_0^x \sin \theta(\tau) d\tau \right).$$
Examples

Concerning **Case 3** in classification theorem, let us choose for example \( c_1 = (0, 1, 0) \), \( c_2 = (0, 0, 1) \) and \( c_3 = (1, 0, 0) \). The parametrization in this case is given by

**Example**

\[
F(x, y) = \left( A(y) \sinh \left( \int_0^x \cos \theta(\tau) d\tau \right) + B(y) \cosh \left( \int_0^x \cos \theta(\tau) d\tau \right), \int_0^x \sin \theta(\tau) d\tau \right)
\]

where \( A(y) = \left( y, 1 - \frac{y^2}{2}, \frac{y^2}{2} \right) \) and \( B(y) = \left( y, -\frac{y^2}{2}, 1 + \frac{y^2}{2} \right) \).
Examples

If $\theta(x) = x^2$, the surface is

Example

$$F(x, y) = \left( A(y) \sinh \left( \sqrt{\frac{\pi}{2}} C\left( \sqrt{\frac{2}{\pi}} x \right) \right) + B(y) \cosh \left( \sqrt{\frac{\pi}{2}} C\left( \sqrt{\frac{2}{\pi}} x \right) \right), \right.$$ 

$$\sqrt{\frac{\pi}{2}} S\left( \sqrt{\frac{2}{\pi}} x \right)$$

where $C$ and $S$ are the traditional notations for the Fresnel integrals $C(z) = \int_0^z \cos \left( \frac{\pi t^2}{2} \right) dt$ respectively $S(z) = \int_0^z \sin \left( \frac{\pi t^2}{2} \right) dt$. The curve involved in the classification theorem is given in this case by $(\phi(x), \chi(x)) = (C(x), S(x))$, known as Cornu spiral.
Minimality

Theorem (Dillen, M., Nistor, 2011)

Let \( M \) be a surface isometrically immersed in \( \mathbb{H}^2 \times \mathbb{R} \), with \( \theta \neq 0, \pi/2 \). Then \( M \) is minimal with \( T \) as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by

\[
F : M \longrightarrow \mathbb{H}^2 \times \mathbb{R}
\]

\[
F(x, y) = \left( \frac{b(x)}{\sqrt{1 + c_1^2 - c_2^2}}, \frac{\sqrt{a^2(x)} + 1}{\sqrt{1 + c_1^2 - c_2^2}} \sinh y, \frac{\sqrt{a^2(x)} + 1}{\sqrt{1 + c_1^2 - c_2^2}} \cosh y, \chi(x) \right)
\]

\[
F(x, y) = \left( \frac{\sqrt{a^2(x)} + 1}{\sqrt{c_2^2 - c_1^2 - 1}} \cos y, \frac{\sqrt{a^2(x)} + 1}{\sqrt{c_2^2 - c_1^2 - 1}} \sin y, \frac{b(x)}{\sqrt{c_2^2 - c_1^2 - 1}}, \chi(x) \right)
\]

\[
F(x, y) = \left( b(x), \frac{b(x)}{2} (1 - y^2) - \frac{1}{2b(x)}, \frac{b(x)}{2} (1 + y^2) + \frac{1}{2b(x)}, \chi(x) \right)
\]
Minimality

Theorem (cont.)

Let $M$ be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then $M$ is minimal with $T$ as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by

where

$$\chi(x) = \int_0^x \frac{1}{\sqrt{a^2(\tau) + 1}} \, d\tau$$

with $a(x) = c_1 \cosh x + c_2 \sinh x$, $b(x) = a'(x)$ and $c_1, c_2 \in \mathbb{R}$. 
Flatness

Theorem (Dillen, M., Nistor, 2011)

Let $M$ be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \neq 0, \frac{\pi}{2}$. Then $M$ is flat with $T$ as principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by $F : M \rightarrow \mathbb{H}^2 \times \mathbb{R}$

\[
F(x, y) = \left( \frac{x}{\sqrt{c + 1}} \cos y, \frac{x}{\sqrt{c + 1}} \sin y, \frac{\sqrt{x^2 + c + 1}}{\sqrt{c + 1}}, \chi(x) \right)
\]

\[
F(x, y) = \left( \frac{\sqrt{x^2 + c + 1}}{\sqrt{-c - 1}}, \frac{x}{\sqrt{-c - 1}} \sinh y, \frac{x}{\sqrt{-c - 1}} \cosh y, \chi(x) \right)
\]

\[
F(x, y) = \left( xy, \frac{x}{2} (1 - y^2) - \frac{1}{2x}, \frac{x}{2} (1 + y^2) + \frac{1}{2x}, \chi(x) \right)
\]

where

\[
\chi(x) = \int_{x}^{x} \frac{\sqrt{\tau^2 + c}}{\sqrt{\tau^2 + c + 1}} d\tau, \quad c \in \mathbb{R}.
\]
The upper half plane model of $\mathbb{H}^2$

Method 1: Use **Cayley transformations** from $\mathcal{H}$ to $H^+$

\[
\begin{align*}
x_1 &= \frac{X}{Y} \\
x_2 &= \frac{X^2 + Y^2 - 1}{2Y} \\
x_3 &= \frac{X^2 + Y^2 + 1}{2Y}.
\end{align*}
\]

\[
\begin{align*}
X &= \frac{x_1}{x_3 - x_2} \\
Y &= \frac{1}{x_3 - x_2}.
\end{align*}
\]

Method 2: Analytical approach - solving the problem in $\mathbb{H}^+$ and then showing the consistence of results with $\mathcal{H}$:

A.I. Nistor, *On a class of surfaces in $\mathbb{H}^+ \times \mathbb{R}$*, preprint 2010.
The upper half plane model of $\mathbb{H}^2$

Method 1: Use Cayley transformations from $\mathcal{H}$ to $\mathbb{H}^+$

\[ x_1 = \frac{X}{Y} \]
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\[ X = \frac{x_1}{x_3 - x_2} \]
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Method 2: Analytical approach - solving the problem in $\mathbb{H}^+$ and then showing the consistence of results with $\mathcal{H}$:

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Canonical coordinates in $\mathbb{E}^3$

The characterization theorem:

**Theorem (M., Nistor, 2011)**

Let $M$ be an isometrically immersed surface in $\mathbb{E}^3$. Let $(x, y)$ be orthogonal coordinates on $M$ such that $T$ is collinear to $\partial_x$. Then, $T$ is a principal direction on $M$ everywhere if and only if $\theta_y = 0$.

Canonical coordinates in $\mathbb{E}^3$

The classification theorem:

**Theorem (M., Nistor, 2011)**

A surface $M$ isometrically immersed in $\mathbb{E}^3$ with $T$ a canonical principal direction is given (up to isometries of $\mathbb{E}^3$) by one of the following cases:

- **Case 1.**

  $$F : M \rightarrow \mathbb{E}^3, \quad F(x, y) = \left( \phi(x)(\cos y, \sin y) + \gamma(y), \int_0^x \sin \theta(\tau) d\tau \right)$$

  where

  $$\gamma(y) = \left( -\int_0^y \psi(\tau) \sin \tau d\tau, \int_0^y \psi(\tau) \cos \tau d\tau \right)$$

- **Case 2. (Cylinders)**

  $$F : M \rightarrow \mathbb{E}^3, \quad F(x, y) = \left( \phi(x) \cos y_0, \phi(x) \sin y_0, \int_0^x \sin \theta(\tau) d\tau \right) + y\gamma_0$$

  where $\gamma_0 = (-\sin y_0, \cos y_0, 0)$, $y_0 \in \mathbb{R}$, $\phi'(x) = \cos \theta$. 

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Theorem (M., Nistor, 2011)

Let $M$ be a surface isometrically immersed in $\mathbb{E}^3$. Then $M$ is a minimal surface with $T$ a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$$F : M \rightarrow \mathbb{E}^3$$

$$F(x, y) = \left( \sqrt{x^2 + c^2} \cos y, \sin y, \ln (x + \sqrt{x^2 + c^2}) \right), \ c \in \mathbb{R}.$$

Remark

Moreover, we notice that this surface can be obtained rotating the catenary around the $Oz$-axis. Hence, we obtain that the only minimal surface in the Euclidean space with a canonical principal direction is the catenoid.
Theorem (M., Nistor, 2011)

Let $M$ be a surface isometrically immersed in $\mathbb{E}^3$. Then $M$ is a flat surface with $T$ a principal direction if and only if the immersion is, up to isometries of the ambient space, given by

$$F : M \to \mathbb{E}^3, \quad F(x, y) = \left( \phi(x) \cos y_0, \phi(x) \sin y_0, \int_0^x \sin \theta(\tau) d\tau \right) + y \gamma_0$$

where $\gamma_0 = (- \sin y_0, \cos y_0, 0), \ y_0 \in \mathbb{R}$. Here $\phi(x)$ represents a primitive of $\cos \theta$.

Notice that this is Case 2. (Cylinders) from the classification theorem.
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