WHITNEY
EXTENSION
THM (1934)
RECALL NOTATION

\[ F \in C^m(\mathbb{R}^n), \quad x \in \mathbb{R}^n \Rightarrow \]

\[ J_x(F) = \text{m}^{th} \text{ degree Taylor poly of } F \text{ at } x \]

\[ J_x(F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y-x)^\alpha. \]

So

\[ J_x(F) \in P \]

\[ P = \left\{ \text{vector space of all (real-valued)} \right\} \]

\[ \left[ \text{polys of degree } \leq m \text{ on } \mathbb{R}^n \right] \]
\[ C^m(\mathbb{R}^n) = \left\{ \text{space of real-valued} \right. \\
\left. \text{functions on } \mathbb{R}^n \right. \\
\text{whose derivatives up to order } m \right. \\
\text{are continuous and bounded} \]
A WHITNEY FIELD

IS A FAMILY

\[ \vec{P} = (P^x)_{x \in E} \]

OF POLYNOMIALS \( P^x \in \mathcal{P} \),

INDEXED BY THE POINTS \( x \in E \),

WHERE \( E \) IS A SUBSET OF \( \mathbb{R}^n \).
Whitney's Question:

Given a Whitney field

$$\vec{p} = (P^x)_{x \in E}$$

with $E \subset \mathbb{R}^n$ compact.

How can we decide whether there exists $F \in C^m(\mathbb{R}^n)$ such that

$$J_x(F) = P^x$$

for all $x \in E$?
Compare with question from Talk I:

Given a function $f : E \to \mathbb{R}$, how can we decide whether there exists $F \in C^m(\mathbb{R}^n)$ such that $F(x) = f(x)$ for all $x \in E$. 
WHITNEY'S THM (1st Version)

For a Whitney field \( \overrightarrow{\mathcal{P}} = (P^x)_{x \in E} \), \( E \) compact, the following are equivalent:

(A) There exists \( F \in C^m(\mathbb{R}^n) \)
    s. t. \( J_x(F) = P^x \) for all \( x \in E \).

(B) For each multi-index \( \alpha \) (\( 1 \leq |\alpha| \leq m \)),
    we have
    \[
    | \partial^\alpha (P^x - P^y)(x) | = o \left( |x-y|^{m-|\alpha|} \right)
    \]
    as \( |x-y| \to 0 \), \( x, y \in E \).
More Precise Version

Let $\mathcal{P} = (P^x)_{x \in E}$ be a Whitney field.

Let $M$ be a non-negative real number.

Suppose:

(A) $|\partial^x P^x (x)| \leq M$ for all $|x| \leq m$, $x \in E$.

(B) $|\partial^x (P^x - P^y) (x)| \leq M |x - y|^{m-1}|x|$

for all $|x| \leq m-1$, $x, y \in E$.

(C) $|\partial^x (P^x - P^y) (x)| = o(|x - y|^{m-1})$

as $|x - y| \to 0$ ($x, y \in E$) for each $|x| \leq m$.

Then $\exists F \in C^m(\mathbb{R}^n)$ s.t.

- $J_x(F) = P^x$ (all $x \in E$)
- $\|F\|_{C^m(\mathbb{R}^n)} \leq CM$ depends only on $m, n$. 
Thus, we can compute the least possible \( \inf \) \( C^m \) norm of a function \( F \) that agrees with a given Whitney field \( \vec{P} = (P^x)_{x \in E} \) up to a constant factor depending only on \( m, n \).
For $C^2(\mathbb{R}^n)$,

(with a better choice of the $C^2$ norm),

LeGruyer & Wells

found an exact formula

for the least possible norm!
WHITNEY'S PROOF GIVES AN EXPLICIT FORMULA FOR $F$.

$F$ depends linearly on $\vec{P}$

If $\vec{P} = (P^x)_x \in \mathcal{E}$, then for any given point $y \in \mathbb{R}^n$, $F(y)$ is determined entirely by $P^x$, $\ldots$, $P^x_k$.

- $k \leq C$ depends only on $m, n$
- $x_1, \ldots, x_k$ depend only on $y$, but not on $\vec{P}$.
\( \vec{p} = (p^x)_{x \in E}, \quad F = T \vec{p} \in C^m(\mathbb{R}^n) \)

\[ F(y) = \sum_{k=1}^{K} \lambda_k(p^{x_k}) \]

\( \lambda_k : P \rightarrow \mathbb{R} \) are linear functionals

\( x_1, \ldots, x_k \in E \)

\( K \leq C \)

The \( \lambda_k \) and \( x_k \) depend on \( y \), but not on \( \vec{p} \).

\( \vec{p} \rightarrow F \) has "bounded depth"
SKETCH OF WHITNEY'S PROOF
MAIN STEPS

- Whitney Cubes
- Whitney Partition of Unity
- The Extension F
- Check that it works
PREPARE TO DEFINE

WHITNEY CUBES

\[ \square \]

\[ \mathcal{Q} \]
\[ \delta_{AQ} = \delta_Q \]

- \( AQ \) and \( Q \) have same center
Bisecting a cube
**Constructing Whitney Cubes**

Given $E \subset \mathbb{R}^n$.

Start with a big cube $Q^0$ containing $E$ in its middle half.

( Say $\delta_{Q^0} = 1024$ )
Construction of the Whitney Cubes

Proceeds in steps:

Step 0,

Step 1,

Step 2,
At each step \( i \)
we produce a partition
of \( Q^0 \) into finitely
many cubes.

The partition at step \( 0 \)
consists of the single
cube \( Q^0 \).
The partition in step \((i+1)\) refines the partition in step \(i\).

To produce the partition of step \((i+1)\), we bisect some of the cubes from the partition in step \(i\).
Rule:

Let $Q$ be a cube of the step $i$ partition.

- If $3Q \cap E = \emptyset$, then we include $Q$ in the step $(i+1)$ partition, and we call $Q$ a Whitney cube.
If $3Q \cap E \neq \emptyset$,
then we bisect $Q$
into its children $Q_1, Q_2, \ldots, Q_{2^n}$,
and include those children (but not $Q$)
in the step $(i+1)$ partition.
Q arises at step i

3Q ∩ E = ∅

⇒ Q survives, and appears at step i + 1

This Q is a Whitney cube
Q arises at Step \( i \),
\[ 3Q \cap E \neq \emptyset \implies \]
At Step \((i+1)\), Q is replaced by its 4 children \( Q_1, \ldots, Q_4 \).
Basic Properties of Whitney Cubes

- The Whitney Cubes form a partition of \( \mathbb{R}^n \setminus E \).

- The diameter of any Whitney cube is comparable to its distance from \( E \).
Proof:

Let $Q = \text{Whitney cube,}$

$Q$ arose as a child of a cube $Q^+.$

We decided to retain $Q,$

but we decided not to retain $Q^+.$

So $3Q \cap E = \emptyset$

but $3Q^+ \cap E \neq \emptyset.$
IF TWO WHITNEY CUBES $Q$ AND $Q'$ TOUCH, THEN $S_Q$ AND $S_{Q'}$ ARE COMPARABLE,

$$\frac{1}{2} S_Q \leq S_{Q'} \leq 2 S_Q$$

ANY GIVEN WHITNEY CUBE $Q$ TOUCHES AT MOST A BOUNDED NUMBER OF OTHER WHITNEY CUBES $Q'$. 
We have defined the Whitney cubes.

Next, we define the Whitney partition of unity.
For each Whitney cube $Q$, fix a smooth function $\tilde{\theta}_Q$ on $\mathbb{R}^n$, satisfying:

- $\tilde{\theta}_Q \geq 0$
- $\tilde{\theta}_Q = 1$ on $Q$
- $\tilde{\theta}_Q$ supported in $(1.01)Q$
- $|\partial^\alpha \tilde{\theta}_Q| \leq C \frac{\delta^{-|\alpha|}}{\delta Q}$ for $|\alpha| \leq m$
Let $\psi = \sum_{Q'} \tilde{\theta}_{Q'}$, where $Q'$ varies over all the Whitney cubes.

If $x \in \text{supp}(\tilde{\theta}_{Q'})$, then $\delta_{Q'} \sim \text{dist}(x, E)$.

In particular, $x \in \text{supp}(\tilde{\theta}_{Q'})$ for at most $C$ distinct Whitney cubes $Q'$.

Therefore, $\psi$ has the following properties.
\[ y = \sum_{Q'} \tilde{\theta}_{Q'} \]

- \( y \geq 1 \) on \( \mathbb{R}^n \setminus E \)

- \( y = 0 \) on \( E \)

- \( |\mathfrak{e}^x \psi(x)| \leq C [\text{dist}(x, E)]^{-1|x|} \)

  For \( |x| \leq m, \ x \in \mathbb{R}^n \setminus E \)

- \( |\mathfrak{e}^x \psi(x)| \leq C \int_Q^{-1|x|} \) on \( \text{supp} \left( \tilde{\theta}_Q \right) \)

- \( \psi(x) \geq 1 \) on \( \text{supp} \left( \tilde{\theta}_Q \right) \)
Now, for each Whitney cube $Q$, we set

$$\tilde{\Theta}_Q = \frac{\tilde{\Theta}_Q}{\tilde{\psi}} = \frac{\tilde{\Theta}_Q}{\sum_{Q'} \tilde{\Theta}_{Q'}}$$
on $Q$

Thus,

$$\sum_{Q} \Theta_{Q} = \begin{cases} 1 & \text{for } x \in Q \cap E \\ 0 & \text{for } x \in E \end{cases}$$

This is the Whitney partition of unity.
From the properties of $\Theta$, we easily derive the basic properties of the $\Theta$. 
BASIC PROPERTIES OF $\Theta_Q$

$\sum_{Q} \Theta_Q(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R}^d \setminus E \\ 0 & \text{for } x \in E \end{cases}$

For each Whitney cube $Q$,

- $\text{supp } (\Theta_Q) \subset (1.01)Q$

and

- $|x^a \Theta_Q(x)| \leq C \int_{Q} \frac{-101}{Q}$ for all $x \in \mathbb{R}^d$, $|x| \leq m$. 

We have now defined the Whitney cubes and the Whitney partition of unity.

Now we can define the function $F$. 
For each Whitney cube $Q$, we pick a point $x(Q) \in E$ as close as possible to $E$.

Recall, we are given a Whitney field

$$\vec{P} = (P^*)_{x \in E}$$

We want a function $F \in C^m(\mathbb{R}^n)$ s.t.

$$J_x(F) = P^*$$ for each $x \in E$.

Will define $F$ on $Q$.
Define

\[ F(x) = \begin{cases} 
\sum_Q \theta_Q(x) P^{x(Q)}_x & \text{for } x \in \mathbb{E} \\
\varphi^x(x) & \text{for } x \in E
\end{cases} \]

Note: \( F \) depends linearly on the map \( \varphi = (\varphi^x)^{x \in E} \) in a very simple way.

The map \( \varphi \mapsto F \) is a linear map of bounded depth.
Now we have defined the function $F$.

Must show:

\[ \{ \text{Hypotheses of Whitney's Thm.} \} \implies \begin{cases} \quad F \in C^m(\mathbb{R}^n) \\ J_x(F) = P_x \quad (x \in \mathbb{E}) \\ \lVert F \rVert_{C^m(\mathbb{R}^n)} \leq C \end{cases} \]
RECALL HYPOTHESES:

- \( |2^x \mathbf{p}_x(\alpha) \| \leq M \) (all \( x \in E \), \( 1 \alpha \leq m \))

- \( |\mathbf{p}_x(\alpha) - \mathbf{p}_y(\alpha)\| \leq M |x-y|^{m-1} \alpha \)
  (\( x, y \in E \), \( 1 \alpha \leq m-1 \))

- \( |\mathbf{p}_x(\alpha) - \mathbf{p}_y(\alpha)\| = o\left(|x-y|^{m-1} \alpha \right) \)
  as \( |x-y| \to 0 \), \( x, y \in E \), \( 1 \alpha \leq m \).
**KEY IDEA**

Let $x \in Q^o \setminus E$.

Say $x \in \hat{Q}$ (WHITNEY CUBE).

Then in a nbd. of $x$, write

$$F = \sum_Q \epsilon_Q P^x(Q)$$

$$= P^x(\hat{Q}) + \sum_Q \epsilon_Q \left[ P^x(Q) - P^x(\hat{Q}) \right]$$
To show that

\[ \| f \|_{C^m(Q^0)} \leq CM, \]

It will be enough

To show that

\[ \forall x \in \{ \theta \in P^x(Q) - P^x(\hat{Q}) \} \text{ where } (x) \leq CM \]

For \( \text{supp} \theta_Q \in X \).
CRUCIAL POINT:

\[ |x(Q) - x(\hat{Q})| \leq Cs_{\hat{Q}} \]

AND

\[ \frac{1}{2} s_{\hat{Q}} \leq s_{Q} \leq 2s_{\hat{Q}} \]

WHEN \( \text{supp} \theta_{Q} \in x \).
To estimate
\[ \Theta \{ \Theta_Q \cdot \left[ P^x(Q) - P^x(\hat{Q}) \right] \} (x) \]
we use the estimates
\[ |e^\beta \Theta_Q (x)| \leq C s_Q^{-|\beta|} \leq C' s_Q^{-|\beta|} \]
for \(|\beta| \leq m\)

and
\[ |e^\beta \left[ P^x(Q) - P^x(\hat{Q}) \right] (x)| \leq C m s_Q^{m-|\beta|} \]
for \(|\beta| \leq m\).
$\delta\{\Theta_Q \cdot [P^x(Q) - P^x(\hat{Q})]\}(x)$

is a sum (over $\beta + \gamma = \alpha$) of terms

$[\delta^\beta \Theta_Q(x)] \cdot [\delta^\gamma (P^x(Q) - P^x(\hat{Q}))(x)]$

Dominated by $S_{\hat{Q}}^{m - 1\beta 1}$

Dominated by $MS_Q^{m - 1\alpha 1}$

Product $\leq MS_Q^{m - 1\alpha 1}$
RECALL,
\[ \hat{A} \subset Q^0 \leftarrow \text{SIDELENGTH 1024} \]

so \[ S_Q \leq 1024. \]

So, for \( |x| \leq m \), we have shown that

\[ \left| E^* \left\{ \theta \in \mathcal{P}^x(Q) \ - \mathcal{P}^x(\hat{A}) \right\} (x) \right| \leq CM. \]

Therefore,

\[ \| F \|_{C^m(Q^0)} \leq CM. \]
Those are the main ideas in the proof of the Whitney Extension Theorem.
Whitney's Theorem tells us when there exists $F \in C^m$ that agrees with a given Whitney field on $E$.

We really want to know whether there exists $F \in C^m$ that agrees with a given function on $E$. 
Although Whitney's THM answers an easier variant of the "real" problem, both the theorem and its proof contain important lessons for us (and for analysis).
Lessons from the Proof of Whitney's Theorem
Lesson 1

* We will be interested in products of the form

\[
[\text{Factor 1}] \cdot [\text{Factor 2}]
\]

Where

\[
|\partial^\beta (\text{Factor 1})(x)| \leq S^{-1}\beta
\]

and

\[
|\partial^\beta (\text{Factor 2})(x)| \leq S^{m-1}\beta
\]

for \( |\beta| \leq m \).
Lesson 2:

Whitney's Theorem for Finite Sets
Given $M > 0$, $\tilde{P} = (P^x)_{x \in E}$, $E$ finite. Assume:

$|\theta^x P^x(x)| \leq M$ for all $x \in E$, $|\alpha| \leq m$.

$|\theta^x (P^x - P^y)(x)| \leq M |x - y|^m |\alpha|$, all $x, y \in E$, $|\alpha| \leq m - 1$.

Then there exists $F \in C^m(\mathbb{R}^n)$ with norm

$\| F \|_{C^m(\mathbb{R}^n)} \leq CM$, such that

$J_x(F) = P^x$ for all $x \in E$.
Note: To decide whether

\[ |\partial^\alpha_{P^x}(x)| \leq M \]

and

\[ |\partial^\alpha(P^x - P^y)(x)| \leq M |x - y|^{m-1} \alpha | \]

we may examine

the quadratic form

\[ Q(P) = \sum_{x \in E} \sum_{|\alpha| \leq m} (\partial^\alpha_{P^x}(x))^2 + \]

\[ \sum_{x, y \in E} \sum_{|\alpha| \leq m-1} \left( \frac{\partial^\alpha(P^x - P^y)(x)}{|x - y|^{m-1} \alpha |} \right)^2 \]
The Quadratic Form $2(\mathbf{F})$

is a useful idea when $\#(E) \leq C$,

but not when $\#(E)$

is arbitrarily large.
APPLICATION

Let \( f : E \to \mathbb{R}, \ E \subset \mathbb{R}^n \) finite.

Define

\[
\|f\|_E = \inf \text{ of } \|F\|_{C^m(\mathbb{R}^n)} \text{ over all } F \in C^m(\mathbb{R}^n) \text{ such that } F = f \text{ on } E.
\]

PROBLEM:

Compute the order of magnitude of

\[
\|f\|_E
\]
Solution in case\( \#(E) \leq C \)

\[ \|f\|_E^2 \text{ is comparable to} \]

The min of the Quadratic Form \( Q(\vec{p}) \)

Over all \( \vec{p} = (P^x)_{x \in E} \)

Such that \( P^x(x) = f(x) \) \( \forall x \in E \).

That’s immediate from Whitney’s Thm for finite sets.
So, computing the order of magnitude of $\|f\|_E$

when $\#(E) \leq C$

is reduced to linear algebra.

(& the matrices are of bounded size.)
Lesson 3

The Calderón-Zygmund Decomposition
Start with a unit cube $Q^0 \subset \mathbb{R}^n$.

If we like it, then keep it;

If we don't like it, then bisect it into subcubes $Q_1, \ldots, Q_2^n$,

& examine each of those in turn.
To examine a cube $Q$,
we ask:

Do we like $Q$?

- If so, then we keep $Q$.
- If not, then we bisect $Q$ into $2^n$ subcubes, and examine each of those subcubes.
Starting with $Q^0$, and repeatedly bisecting as prescribed above, we arrive at a collection of pairwise disjoint subcubes of $Q^0$. They are the Calderón-Zygmund cubes.
How do we decide whether we like a cube \( Q \)?

**Whitney**: We like \( Q \) if \( 3Q \cap E = \emptyset \).

**Calderón & Zygmund** (1950's; see also Marcinkewicz, 1930's)

We like \( Q \) if

\[
\frac{1}{\text{vol}(Q)} \int_Q |f(x)| \, dx > \lambda
\]

for given fn. \( f \) & number \( \lambda \).
We can give any rule we please.

For each Calderón-Zygmund cube $Q$, we know that

We like $Q$,

but

$Q$ arises as a child of another cube $Q^+$, and we don't like $Q^+$. 

67
The Calderón-Zygmund cubes are pairwise disjoint.

If we like every sufficiently small cube, then the Calderón-Zygmund cubes partition $Q^0$. 
The Calderón–Zygmund decomposition is a very important idea, with many applications.

We will use a particular C-Z decomposition to prove our main results.
AN APPLICATION OF THE WHITNEY CUBES

We will construct the "WELL-SEPARATED PAIRS" DECOMPOSITION FROM TALK 1
RECALL:

GIVEN $E \subset \mathbb{R}^n$, $N = \#(E) < \infty$.

WANT TO PARTITION

$E \times E \setminus$ DIAGONAL

INTO CARTESIAN PRODUCTS $E'_\nu \times E''_\nu$ ($\nu = 1, 2, \ldots, \nu_{\text{max}}$)

WITH

$\nu_{\text{max}} \leq CN$

$diam(E'_\nu) + diam(E''_\nu) \leq 10^{-3} \cdot \text{dist}(E'_\nu, E''_\nu)$
CONSTRUCTION

OF THE

WELL-SEPARATED PAIRS

DECOMPOSITION
Let $Q^0 \supset E$ be a large cube.

We perform a Whitney decomposition of $Q^0 \times Q^0 \setminus \text{Diagonal}$.

We start with the cube $Q^0 \times Q^0$, and we stop bisecting the cube $Q' \times Q''$ whenever $\text{diam}(Q') + \text{diam}(Q'') = 10^{-3} \text{dist}(Q', Q'')$. 
Let \( Q'_v \times Q''_v \) \((v=1, \ldots, v_{\text{MAX}})\) be the Whitney cubes that intersect \( E \times E \).

Our Cartesian products \( E'_v \times E''_v \) are simply the sets
\[
(E \times E) \cap (Q'_v \times Q''_v)
\]
These $E_{\nu'} \times E_{\nu''}$ are clearly well-separated.

One can show that $\nu_{\text{max}}$ (the number of distinct $E_{\nu'} \times E_{\nu''}$) is at most $C \cdot N$

(We won't do that in these talks.)
Callahan & Kosaraju

give a simple,
elegant, efficient
algorithm to
compute a
well-separated pairs
decomposition.

We won't do that in
these talks.
I hope this conveys some hint of the power of the Whitney/Caldern-Zygmund decomposition. We'll see more in the later talks.