

Spring Lecture Series N.33

Partial Differential Equations in Conformal Geometry (2008)

Principal Lecturer: Sun-Yung Alice Chang, Princeton University

Public Lecture by Frank Morgan, Williams College, "Soap Bubbles and Mathematics"

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Participants

Invited Speakers:

Ivana Alexandrova (East Carolina University)

Title: Semi-Classical Resolvent Estimates at the Maximum of the Potential

Abstract: We prove upper and lower bounds on the resolvent of the Schrodinger operator at an energy, which is the global non-degenerate maximum of the potential. Our upper bound improves the earlier bound of Nakamura '91 and our lower bound shows that our upper bound is near sharp. We next discuss an application of our upper resolvent estimate to proving an asymptotic expansion of the total scattering cross-section.

Chiun-Chuan Chen (Academia Sinica, Taiwan)

Title: Topological degree for a Liouville type equation with singular source

Abstract: We consider a Liouville type equation in two dimension domains which arises from prescribing Gaussian curvature problem, the mean field limit of vortices in Euler flows and limit cases of Chern-Simons models. The "total mass", which equals the integral of the nonlinear term, plays a key role for this equation. The solutions can blow up when the total mass tends to some critical values. We will discuss how the delta functions in the source term affect the blowup behavior of the solutions and present a recently-discovered combinatorial formula of the Leray-Schauder degree for the problem.

Sophie Chen (UC Berkeley)

Title: Conformally compact Einstein manifolds

Abstract: Conformally compact Einstein manifolds are models of Euclidean formulation of Lorenzian manifolds in general relativity. In this talk, we will study the subject from mathematical point of view and present a compactness result under natural conformal invariant conditions.

Maria Del Mar Gonzalez (U. Politecnica de Catalunya, Spain)

Title: Some problems involving fractional order operators

Abstract: I am interested in the study of fractional-order non-local operators. They appear when modeling phase transitions or crystal dislocations, for instance, but they can also be studied from the conformal geometry point of view. The common tool in all these settings is a characterization of the operator through an extension problem to a manifold of one higher dimension.

Jose Espinar (University of Granada, Spain)

Title: Conformally invariant equations via hypersurface theory in hyperbolic space

Abstract: we provide a bridge between the theory of conformal metrics on the sphere, and hypersurface theory in hyperbolic space. In particular, we show that the Nirenberg problem is essentially equivalent to a natural formulation of the Christoffel problem in the hyperbolic setting, and that several important problems regarding the Schouten tensor of conformal metrics admit a hypersurface theory interpretation. As a consequence, we obtain several uniqueness results among Weingarten hypersurfaces for spheres and horospheres in hyperbolic space.

Robin Graham (U. Washington)

Title: The ambient metric beyond the obstruction in even dimensions

Abstract: The ambient metric construction in conformal geometry will be recalled, and also the equivalent formal construction of asymptotically hyperbolic Poincare-Einstein metrics. A modification of the construction in even dimensions will be described which results in a family of smooth infinite order generalized ambient and Poincare metrics. The generalized ambient metrics can be used to extend conformal invariant theory to all orders in even dimensions. This is joint work with Kengo Hirachi.

Matthew Gursky (U. Notre Dame)

Title: A Monge-Ampere equation in conformal geometry

Abstract: We give some background on the problem of prescribing the determinant of the conformal Schouten tensor, a fully nonlinear PDE. Since ellipticity of the equation turns out to be a strong geometric constraint, we introduce a perturbed version which shares many of the same analytic/geometric features, but is elliptic on any conformal manifold. We then give some existence results and applications.

Emmanuel Hebey (U. Cergy-Pontoise, France)

Title: Stability for local low energy solutions of critical elliptic systems

Abstract: We discuss stability for local low energy solutions of critical elliptic systems of Schroedinger equations in potential form. We prove a sharp result which takes into account both the underlying conformal structure and the system structure of the equations.

Guanghao Hong (U. of Iowa)

Title: Mollification of Domains

Abstract: We provide a good idea to mollify non-smooth domains, especially Reifenberg flat domains. That is we mollify the characteristic function of the domain and then take the $1/2$ level set of the mollified function.

Shilong Kuang (U. California Riverside)

Title: A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow

Abstract: We (joint with Qi S. Zhang) establish a point-wise gradient estimate for \forall positive solutions of the conjugate heat equation. This contrasts to Perelman's point-wise gradient estimate which works mainly for the fundamental solution rather than all solutions.

Enrico LeDonne (Yale)

Title: Rigid properties for metric structures on manifolds

Abstract: In this talk, we focus on the rigidity of certain non-smooth metric structures on manifolds. A well-known theorem of Berestovskii states that a finite dimensional geodesic metric space with transitive isometry group is isometric to a homogeneous space G/H equipped with a Finsler-Carnot-Caratheodory metric; here G is a connected Lie group and H is a closed subgroup. Any such metric is defined by a bracket generating sub-bundle of the tangent bundle and a norm on the sub-bundle. We present the problem of describing biLipschitz homogeneous geodesic manifolds, i.e., path metric spaces which are homeomorphic to manifolds and have a transitive group of biLipschitz homeomorphisms. In the discussion it will be crucial the fact that the group of biLipschitz maps, unlikely the isometry group, is not locally compact. We can show that a biLipschitz homogeneous geodesic manifold has to be biLipschitz equivalent to one of the above homogeneous spaces under the assumption that there is a locally compact group acting transitively by biLipschitz maps.

Niels Martin Moller (U. of Aarhus, Denmark)

Title: Extremal metrics for conformal functionals

Abstract: I will talk about such Riemannian metrics on closed n -manifolds. The main examples will be 1) determinants of (integer powers) of conformally covariant operators such as conformal and spinor Laplacians (for n odd), and 2) the total Q -curvature (for n even). As functionals on the space of all Riemannian metrics on the manifold, they satisfy a conformal invariance property. Under arbitrary changes of the metric, one generally studies the Hessian as an operator, and uses ellipticity theory to prove extremality in almost all directions. On spheres, we can classify such Hessians using representation theory for the conformal group, and this allows proving true local extremality at the standard round metric, extending previous results by Kate Okikiolu on determinants of conformal Laplacians. This work is partly joint with Bent Orsted.

Kate Okikiolu (U. of C. San Diego)

Title: The total wavelength of a surface

Abstract: The frequencies at which a surface can vibrate are given by the eigenvalues of the Laplace-Beltrami operator, and the wavelength of the sound produced is inversely proportional to the frequency. The sum of the wavelengths is infinite, but this sum can be regularized. In this talk we discuss this total wavelength, its relationship to the ADM mass from general relativity and other quantities, and its extremal behavior.

Jie Qing (U. of C. Santa Cruz)

Title: On a conformal gap and finiteness theorem for a class of four manifolds

Abstract: In this talk we develop a bubble tree structure for a degenerating class of Riemannian metrics satisfying some global conformal bounds on compact manifolds of dimension 4. Applying the bubble tree structure, we establish a gap theorem, a finiteness theorem for diffeomorphism type for this class. We will also apply the bubble tree structure to the study of the solutions of the class of conformally invariant fully nonlinear equations on a degenerating family of Bach flat metrics.

John Ryan (U. Arkansas)

Title: Sharp L^2 inequalities for Dirac type operators

Abstract: We use the spectrum of a Dirac-Beltrami operator on the sphere to obtain sharp L^2 inequalities for Dirac type on the sphere. These operators include the conformal Laplacian and Paenitz operator. Stereographic projection is used to obtain similar inequalities in euclidean space. Time permitting a p -Dirac operator in euclidean space and on the sphere will also be introduced. This is joint work with Alexander Balinsky (Cardiff).

Yujen Shu (U. C. Santa Barbara)

Title: Einstein warped product

Abstract: We start from an observation that the Einstein condition on a warped product can be reduced to an equation on the base. An Einstein warped product is trivial if the warping function is constant. There are non-trivial Einstein warped products even when they are compact. Here we give several natural conditions on the curvatures of the base that characterize trivial Einstein Warped products.

Neil Trudinger (Aust. Nat. University, Australia)

Title: Hessian and Monge-Ampere type equations with supplementary ellipticity

Abstract: We present a selection of recent results pertaining to Hessian and Monge-Ampere equations, where the Hessian matrix is augmented by a matrix valued lower order operator. Equations of this type arise in conformal geometry, geometric optics and optimal transportation. In particular we will discuss structure conditions, due to Ma, Wang and myself, which imply the regularity of solutions. These conditions are a refinement of a condition used originally by Pogorelev for general equations of Monge-Ampere type in two variables and called strong ellipticity by him.

Shihshu Walter Wei (University of Oklahoma)

Title: Some recent progress in p -harmonic geometry

Abstract: We will study some recent progress in p -harmonic geometry, and its relation with conformal geometry. Some partial differential equations in equivariant differential geometry and in Finsler geometry will be discussed.

Jie Xiao (Memorial University of Newfoundland, Canada)

Title: Chern-Gauss-Bonnet and Isoperimetric Inequalities for Conformal Metrics on R^n , $n \geq 3$

Abstract: In this talk we will report how to use both Q and scalar curvatures to establish Chern-Gauss-Bonnet integral inequalities and their induced isoperimetric deficits and inequalities for conformal metrics on Euclidean spaces with dimension greater than two.

Yu Yuan (U. Washington)

Title: Special Lagrangian equations

Abstract: We survey some recent results on Hessian, gradient estimates, regularity, and global rigidity for special Lagrangian equations with certain convexity. The gradient graph of the solutions are minimal Lagrangian submanifolds in Euclidean space. The special Lagrangian equations in the Pseudo-Euclidean setting are just Monge-Ampere equations, for which one has the corresponding classic results and counterexamples.

Lei Zhang (U. Alabama-Birmingham)

Title: Approximation theorems on prescribing Q-curvature equation on Riemannian manifolds

Abstract: For a fourth order prescribing Q-curvature type equation defined on four dimensional compact Riemannian manifolds, we obtain sharp estimates on the asymptotic behavior of blow-up solutions near their blow-up points. This is a joint work with Gilbert Weinstein.