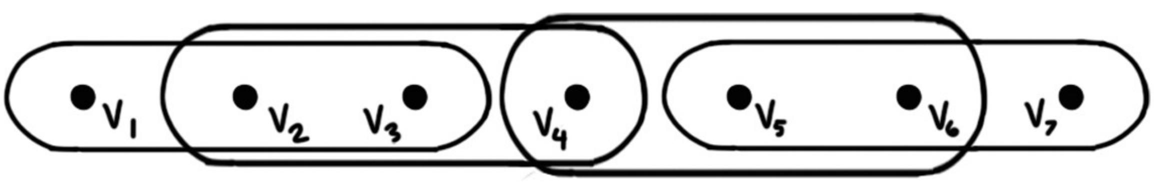
Thesis Proposal:

On 3-Uniform 4-Path Decompositions of Complete 3-Uniform Hypergraphs

Hypergraph decomposition is a relatively new area of research that developed naturally from questions in combinatorics. A common type of combinatorial problem involves partitioning classes of incidence structures into isomorphic copies of a smaller structure. In graph theory, this type of research is known as graph decomposition. That is, we search for ways to decompose the complete graph of order v, denoted K_v, into isomorphic copies of a particular smaller graph, denoted H. The problem of determining all values of v for which an H-decomposition of K_v exists is known as the spectrum problem for H

A graph H is defined as an ordered pair ($V(H)$, $E(H)$), where $V(H)$ is the set of elements called vertices and $E(H)$ is a set of 2-element subsets of $V(H)$ called edges. $|V(H)|$ is called the order of H and $|E(H)|$ is called the size. Hypergraphs are generalizations of graphs. If every edge contains exactly k vertices, we call the hypergraph k-uniform. Thus a graph can also be called a 2-uniform hypergraph. For my thesis, I am interested in decomposing the complete 3-uniform hypergraph of order v into isomorphic copies of my particular graph, denoted as $P_7 = (\{v_1, v_2, \dots, v_7\},\{\{v_1, v_2, v_3\},\{v_2, v_3, v_4\},\{v_4, v_5, v_6\},\{v_5, v_6, v_7\}\})$ and shown below:



P_7 is a 3-uniform 4-path of order 7. This means there are four edges and seven vertices, with exactly three vertices per edge. My methods are based on several existing results, most notably what I will refer to as the Fundamental Lemma for hypergraph decomposition. The Fundamental Lemma provides the necessary conditions for finding an H-decomposition of K_v, based on the fact that the number of edges in H must divide the number of the edges in K_v, which follows as the edges in the isomorphic copies of H partition the edges in K_v. Prior research says that these necessary conditions are asymptotically sufficient as v approaches infinity [5]. As I expect the same to be true for P_7, my thesis will be significant to the developing area of hypergraph decomposition as it will settle the spectrum problem for a currently unsettled hypergraph and contribute to the library of known decompositions.

The method that I will use to settle the spectrum problem for P_7 will be searching for patterns by drawing graphs and calculating the edge lengths. This will be done manually, with some assistance from python code to generate lists of difference classes. Difference classes are groupings used to partition all possible edges in a complete graph based on the distances between the vertices. The set of edges contained in a given difference class is called the orbit of that difference class. For example, the edge incident with the vertices *0, 2,* and *8* in K_9 would be in the difference class of (*1*,*2*,*6*). Such a grouping is obtained by finding the difference between each vertex modulo 9 and arranging them lexicographically. I will use a spreadsheet to track the edges and ensure that each complete graph has been fully decomposed. For efficiency, decompositions will be done cyclically or r-pyramidally whenever possible.

Cyclic decomposition occurs when applying the isomorphism $i \mapsto i+1$ to $V(H)$ preserves the difference classes of the edges in *H*. This requires the difference classes to all contain exactly the same number of edges. With this requirement, we can choose one representative edge from each difference class and obtain the ($v-$¿1) remaining edges in each difference class by applying the aforementioned isomorphism, a process we call clicking.

Occasionally we encounter short orbits, which is when certain difference class(es) contain fewer edges than the others. We use fixed points to force difference classes of the same size. We denote fixed points as ∞_i and note that $\infty_i \mapsto \infty_i$ when we click it. For example, K_{12} contains 220 edges, which are partitioned into eighteen difference classes of size 12 and one difference class of size 4. If we let $V(K_{12})=\mathbb{Z}_{11}\cup\{\infty\}$, then our 220 edges are partitioned into twenty difference classes of size 11. Since cyclic decomposition is the most efficient way to find a decomposition, fixed points allow us to find a similar structure called r-pyramidal decomposition when cyclic decomposition is not possible. R-pyramidal decompositions are obtained in the same fashion as cyclic ones, but using one or more fixed points.

The Fundamental Lemma tells us that the values of *v* for which a P_7-decomposition of K_v exists are $v \equiv 0,1,2,4,6$ (mod 8) [4]. We can break this down by representing the complete graph K_v as ($\frac{v}{8}$¿ groups of 8 vertices, with every possible edge connecting these groups. We can either have an edge contained completely in one grouping, between two groupings, or between three groupings. In the case that $v \equiv 1,2,4,6 \pmod 8$, we also have free vertices that do not complete a full grouping. Thus, we must also account for the edges between loose vertices with one or two groupings, as well as those contained completely in the loose vertices. This method of building K_v allows us to find several smaller structures that together make up the complete graph.

My thesis will also explain methods of “building” necessary pieces to decompose the complete graph. Hypergraphs of smaller size are easier to deal with as there are fewer possible combinations of edges, with fewer copies of the graph needed. We can further decompose the necessary pieces to settle the spectrum problem to work more efficiently. For example, if we need to find a decomposition of the complete tripartite graph $K_{8,8,8}$, we can instead find $K_{4,4,4}$ and automatically obtain the former. Using these tools, in my thesis I will attempt to settle the spectrum problem for P_7.

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[4] H. Hanani, On quadruple systems, *Canad. J. Math.* 12 (1960), 145-157.

[5] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pages.